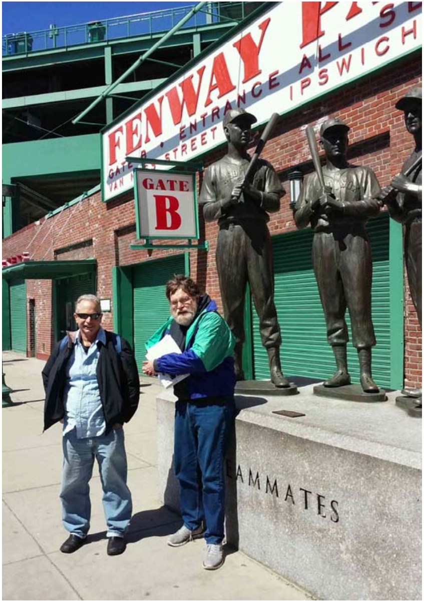
Existence and Regularity of $\mathcal{A}$ Capacitary functions in a Minkoswki Inspired Geometric Problem

John Lewis

Workshop on Real Harmonic Analysis and its Applications to Partial Differential Equations and Geometric Measure Theory
On the Occasion of Steve Hofmann's 60 th Birthday

Happy Birthday Steve



Notation and Definitions

Let $x = (x_1, \ldots, x_n)$ denote a point in Euclidean n space, $\mathbb{R}^n$, $n \geq 2$, with norm, $|x|$. Put

$$B(z, \rho) = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |z - y| < \rho\} \text{ whenever } z \in \mathbb{R}^n,\ \rho > 0,$$

and let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{R}^n$. Set $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and let dx denote Lebesgue n-measure on $\mathbb{R}^n$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q < \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions h with distributional gradient $\nabla h = (h_{x_1}, \ldots, h_{x_n})$, both of which are q th power integrable on O. Let $\|h\|_{1,q} = \|h\|_q + \| \, |\nabla h| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let $\mathcal{H}^\lambda$, $\lambda > 0$, denote λ dimensional Hausdorff measure on $\mathbb{R}^n$.

For fixed $p > 1, \delta \in (0, 1)$, introduce vector fields $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ of p Laplace type satisfying: $\mathcal{A} = \mathcal{A}(\eta)$ has continuous partial derivatives in $\eta_k, 1 \leq k \leq n$, and whenever $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n \setminus \{0\}$:

$$(i)\ \delta|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta)\xi_i\xi_j \text{ and } \sum_{i=1}^{n} |\nabla \mathcal{A}_i(\eta)| \leq \delta^{-1}|\eta|^{p-2} \tag{1}$$

$$(ii)\ \mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\eta/|\eta|).$$

We say that u is $\mathcal{A}$-harmonic in an open set O provided $u \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int \langle \mathcal{A}(\nabla u(y)), \nabla \theta(y) \rangle \, dy = 0 \quad \text{whenever } \theta \in W_0^{1,p}(G). \tag{2}$$

As a short notation for (2) we write $\nabla \cdot \mathcal{A}(\nabla u) = 0$.

An important special case for us is when

$$\frac{\partial \mathcal{A}_i(\eta)}{\partial \eta_j} = \frac{\partial \mathcal{A}_j(\eta)}{\partial \eta_i} \text{ for all } \eta \in \mathbb{R}^n \setminus \{0\} \text{ and } 1 \leq i, j \leq n.$$

Equivalently, for some $f \in C^2(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree p :

$$\mathcal{A}(\eta) = \mathbb{D}f(\eta) = \left(\frac{\partial f}{\partial \eta_1}, \frac{\partial f}{\partial \eta_2}, \ldots, \frac{\partial f}{\partial \eta_n}\right). \tag{3}$$

If $f(\eta) = p^{-1}|\eta|^p$ in (3), then (2) becomes $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ (the so called p Laplace equation).

Observe that solutions remain solutions under translation and dilation but not necessarily under rotations. Also $v = 1 - u$ is a solution to $\nabla \cdot \tilde{A}(\nabla v) = 0$, where $\tilde{A}(\eta) = \tilde{A}(-\eta)$.

Let $E \subset R^n$, $n \geq 2$, be a compact convex set and let $\Omega = \mathbb{R}^n \setminus E$. Using results in Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Dover Publications, 2006, as well as Sobolev type limiting arguments, one can show that given

$p, 1 < p < n$, there exists a unique continuous function $u, 0 < u \leq 1$, on $\mathbb{R}^n$ satisfying

$$\begin{aligned} &(a)\ u \text{ is } \mathcal{A}-\text{harmonic in } \Omega, \\ &(b)\ u \equiv 1 \text{ on } E, \\ &(c)\ |\nabla u| \in L^p(\mathbb{R}^n) \text{ and } u \in L^{p^*}(\mathbb{R}^n) \text{ for } p^* = \frac{np}{n-p}. \end{aligned} \tag{4}$$

if and only if $\mathcal{H}^{n-p}(E) = \infty$. We put

$$\mathrm{Cap}_{\mathcal{A}}(E) = \int_\Omega \langle \mathcal{A}(\nabla u), \nabla u \rangle \, dy$$

and call $\mathrm{Cap}_{\mathcal{A}}(E)$, the $\mathcal{A}-$*capacity of* E while u is the $\mathcal{A}-$ capacitary function corresponding to E in Ω. We note that this definition is a slight extension of the usual definition of "capacity ". However in case,

$$\mathcal{A}(\eta) = p^{-1} \mathbb{D} f(\eta) \text{ on } \mathbb{R}^n \setminus \{0\}$$

then using p homogeneity of f and Euler's formula one gets the usual definition of capacity relative to f. That is,

$$\mathrm{Cap}_{\mathcal{A}}(E) = \inf\left\{ \int_{\mathbb{R}^n} f(\nabla\psi(y))dy \ : \psi \in C_0^\infty(\mathbb{R}^n) \text{ with } \psi \geq 1 \text{ on } E \right\}.$$

If $f(\eta) = p^{-1}|\eta|^p$ one obtains the so called p capacity of E, denoted $\mathrm{Cap}_p(E)$. From the structure assumptions on $\mathcal{A}$ in (1) (i) it follows that

$$c^{-1}\mathrm{Cap}_p(E) \leq \mathrm{Cap}_{\mathcal{A}}(E) \leq c\,\mathrm{Cap}_p(E) \tag{5}$$

From uniqueness of u in (4) we note for $z \in \mathbb{R}^n, \rho > 0$, that if $\tilde{E} = \rho E + z$, then

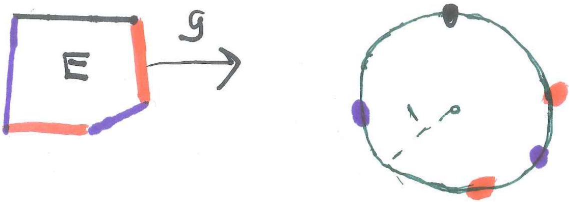
$$(a')\ \mathrm{Cap}_{\mathcal{A}}(\rho E + z) = \rho^{n-p}\,\mathrm{Cap}_{\mathcal{A}}(E),$$
$$(b')\ \tilde{u}(x) = u((x-z)/\rho), x \in \mathbb{R}^n \setminus \tilde{E},\ \text{ is the } \mathcal{A}\text{−capacitary function for } \tilde{E}. \tag{6}$$

So for example, if $z \in \mathbb{R}^n$, $R > 0$,

$$\mathrm{Cap}_{\mathcal{A}}(B(z,R)) = c_1 R^{n-p} \text{ where } c_1 \text{ depends only on } p, n, \delta.$$

On a Minkowski Type Problem for Nonlinear Capacitary Functions

Let $E \subset \mathbb{R}^n$ be a compact convex set with nonempty interior. Then for almost every $x \in \partial E$, with respect to $\mathcal{H}^{n-1}$ measure, there is a well defined outer unit normal, $g(x)$ to ∂E. The function $g : \partial E \to \mathbb{S}^{n-1}$ (whenever defined), is called the Gauss map for ∂E.



The problem originally considered by Minkowski states:

Given a positive finite Borel measure μ on $\mathbb{S}^{n-1}$ satisfying

$$\begin{aligned} &(i) \quad \int_{\mathbb{S}^{n-1}} |\langle \theta, \zeta \rangle| \, d\mu(\zeta) > 0 \text{ for all } \theta \in \mathbb{S}^{n-1}, \\ &(ii) \quad \int_{\mathbb{S}^{n-1}} \zeta \, d\mu(\zeta) = 0, \end{aligned} \tag{7}$$

show there exists up to translation a unique compact convex set E with nonempty interior and

$$\mathcal{H}^{n-1}(g^{-1}(\beta)) = \mu(\beta) \text{ whenever } \beta \subset \mathbb{S}^{n-1} \text{ is a Borel set.} \tag{8}$$

Minkowski, in Volumen und Oberfläche, Math. Ann. **57** (1903), no. 4, 447–495 proved existence and uniqueness of E when μ is discrete or has a continuous density. The general case was treated by Alexandrov in On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies, Mat. Sb. (N.S.), **3** (1938), 27-46. and On the surface area measure of convex bodies, Mat. Sb. (N.S.), **6** (1939), 167-174.

Similar results were obtained by Fenchel and Jessen in Mengenfunktionen und konvexe Körper, Danske Vid. Selsk, Mat.-Fys. Medd. **16** (1938), 1-31.

Jerison in A Minkowski problem for electrostatic capacity, Acta Mathematica, **175** (1996), no. 1, 1–47
considered the following problem: Given $E \subset \mathbb{R}^n$, $n \geq 3$, a compact convex set with nonempty interior let u be the Newtonian or 2 capacitary function for E. Then u is harmonic in $\Omega = \mathbb{R}^n \setminus E$ and from work of Dahlberg in
Estimates of harmonic measure, Arch. Rational Mech. Anal., **65** (1977), no. 3, 275–288, it follows that for $\mathcal{H}^{n-1}$ every $x \in \partial E$,

$$\lim_{y \to x} \nabla u(y) = \nabla u(x) = |\nabla u(x)| \, \nu(x) \text{ nontangentially} \tag{9}$$

where $\nu(x)$ is the unit inner normal to E. Also,

$$\int_{\partial E} |\nabla u|^2 d\mathcal{H}^{n-1} < \infty.$$

If μ is a positive finite Borel measure on $\mathbb{S}^{n-1}$ satisfying (7) then it was shown by Jerison that there exists E a compact convex set having nonempty interior and corresponding Newtonian capacitary function u with

$$\int_{g^{-1}(\beta)} |\nabla u|^2(x)\, d\mathcal{H}^{n-1}x = \mu(\beta) \tag{10}$$

whenever $\beta \subset \mathbb{S}^{n-1}$ is a Borel set and $n \geq 4$. If $n = 3$, there exists a compact convex set E and $b \in (0, \infty)$ for which (10) holds with μ replaced by $b^{-1}\mu$. Moreover he used the Hadamard Variational Formula and the case of equality for Newtonian capacity in a Brunn - Minkowski inequality to show that if $n \geq 4$, then E is the unique set up to translation for which (10) holds, whereas if $n = 3$, then b is unique and E also is unique up to translation and dilation. Jerison's result was generalized by Colesanti, Nyström, Salani, Xiao, Yang, and Zhang (abbreviated CNSXYZ from now on) in
The Hadamard variational formula and the Minkowski problem for p-capacity, Adv. Math. **285** (2015), 1511–1588

To state their result, let E be a compact convex set with nonempty interior, p fixed, $1 < p < n$, and let u be the p capacitary function for E. Then from results of Lewis and Nyström in Boundary behaviour for p harmonic functions in Lipschitz and starlike Lipschitz ring domains, Ann. Sci. École Norm. Sup. (4), **40** (2007), no. 5, 765-813 it follows that (9) holds for u and

$$\int_{\partial E} |\nabla u|^p d\mathcal{H}^{n-1} < \infty.$$

The authors show that if μ is a positive finite Borel measure on $\mathbb{S}^{n-1}$ having no antipodal point masses (i.e , it is not true that $0 < \min\{\mu(\{x\}), \mu(\{-x\})\}$ for some $x \in \mathbb{S}^{n-1}$) and if (7) holds, then for $1 < p < 2$, there exists E a compact convex set with nonempty interior and corresponding p capacitary function u with

$$\int_{g^{-1}(\beta)} |\nabla u|^p(x)\, d\mathcal{H}^{n-1}x = \mu(\beta) \qquad (11)$$

whenever $\beta \subset \mathbb{S}^{n-1}$ is a Borel set.

Moreover assuming the existence of an E for which (11) holds when p is fixed, $1 < p < n$, these authors use the case of equality in the Brunn Minkowski inequality for p capacities to show that E is unique up to translation when $p \neq n - 1$ and unique up to translation and dilation when $p = n - 1$.

Murat Akman, Jasun Gong, Jay Hineman, John Lewis, Andy Vogel (abbreviated AGHLV) have considered an analogous problem for $\mathcal{A} = \nabla f$ capacities in

The Brunn Minkowski problem and a Minkowski Problem for Nonlinear Capacities, submitted in September 2017 (see arxiv:1709.00447 for a preprint).

We prove the following theorem.

Theorem 1

Let μ be a positive finite Borel measure on $\mathbb{S}^{n-1}$ satisfying (7). Fix $p, 1 < p < n$, and let $\mathcal{A} = \nabla f$ be as in (3). If $p \neq n-1$, there exists a compact convex set E with nonempty interior and corresponding $\mathcal{A}$ capacitary function u satisfying

$$
\begin{aligned}
(a) &\quad \text{(9) holds for } u \text{ and } \int_{\partial E} f(\nabla u)\, d\mathcal{H}^{n-1} < \infty. \\
(b) &\quad \int_{g^{-1}(\beta)} f(\nabla u)\, d\mathcal{H}^{n-1} = \mu(\beta) \text{ whenever } \beta \subset \mathbb{S}^{n-1} \text{ is a Borel set.} \\
(c) &\quad E \text{ is the unique set up to translation for which } (b) \text{ holds .}
\end{aligned}
\tag{12}
$$

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If $p = n - 1$ then there exists a compact convex set E with nonempty interior, a constant $b \in (0, \infty)$ and corresponding $\mathcal{A}$-capacitary function u satisfying (9) and

$$(d) \quad b \int_{\mathbf{g}^{-1}(K)} f(\nabla u)\, d\mathcal{H}^{n-1} = \mu(K) \text{ whenever } K \subset \mathbb{S}^{n-1} \text{ is Borel.}$$

(e) E is the unique set up to translation and dilation satisfying (d)

Comments on the Proof of Theorem 1

As a broad outline of the proof in AGHLV we follow CNSXYZ (who in turn used many ideas from Jerison). However, several important arguments in this paper used results from Lewis - Nyström in various papers concerning p harmonic functions vanishing on the boundary of a Lipschitz domain. Thus we first needed to extend these arguments to $\mathcal{A} = \nabla f$ harmonic functions, vanishing on a portion of the boundary of a Lipschitz domain. Following Jerison or CNSXYZ we used these results to first derive a Hadamard variational formula for $\mathcal{A} = \nabla f$ capacitary functions in smooth convex domains and then taking limits show this formula holds in arbitrary convex domains with nonempty interior.

After we have proved all these results we consider a minimum problem similar to the one considered in Jerisons paper and CNSXYZ. However unlike CNSXYZ we are able to show that compact convex sets of dimension $k \leq n-1$ (so with empty interior) cannot be a solution to our minimum problem. To rule out these possibilities we use arguments from Lewis and Nyström in Quasi-linear PDE and low-dimensional sets, to appear in Jems, when $k < n-1$ while if $k = n-1$ we use an argument of Venouziou and Verchota in The mixed problem for harmonic functions in polyhedra of $\mathbb{R}^3$, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math.,**79** (2008), Amer. Math. Soc.

Work in Progress

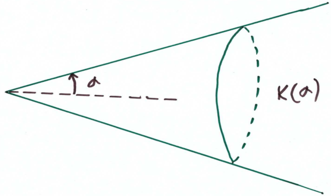
Let E be a compact convex set with nonempty interior and u the corresponding capacitary function for Newtonian capacity. Let μ be the measure on $\mathbb{S}^{n-1}$ corresponding to u in (10). Then Jerison in the paper mentioned earlier also showed that if

$$d\mu = \tau dH^{n-1} \text{ where } \tau > 0 \in C^{k,\beta}(\mathbb{S}^{n-1}), \beta \in (0,1), k = 0, 1, 2, \ldots \tag{13}$$

then Ω is $C^{k+2,\beta}$. CNSXYZ generalized this result to the p capacitary setting when $1 < p < 2$. Both papers make important use of estimates for solutions with vanishing boundary values to their PDE's in cones. This led us (Murat Akman, myself, and Andrew Vogel) to investigate what we now call

An Eigenvalue Problem for Nonlinear PDE of p Laplace type.

To discuss this problem given $x \in \mathbb{R}^n \setminus \{0\}$ introduce spherical coordinates $r = |x|, x_1 = r\cos\theta, 0 \le \theta \le \pi$ and put



$K(\alpha) = \{x \in \mathbb{R}^n : r > 0, 0 < \theta < \alpha\}, \alpha \in (0, \pi]$.
Fix $p, 1 < p < \infty, \alpha \in (0, \pi]$ and let $u > 0$ be $\mathcal{A}$ harmonic in $K(\alpha)$ with continuous boundary value 0 on $\partial K(\alpha)$ and $u(1, 0 \ldots, 0) = 1$. Using arguments from Lewis, Lundström, and Nyström in
Boundary Harnack Inequalities for Operators of p Laplace Type in Reifenberg Flat Domains, Proceedings of Symposia in Pure Mathematics **79** (2008), 229-266,
one can show that If $p \in (1, \infty)$ and $\alpha \in (0, \pi)$, then u exists, is unique, and of the form

$$u(x) = r^\lambda \phi(\theta), \ \ r > 0, \ 0 \leq \theta < \alpha, \ \text{with } \phi(0) = 1, \phi(\alpha) = 0 \tag{14}$$

where $\lambda \in (0, \infty)$. This statement is also true for $\alpha = \pi$ if $n - 1 < p$.

Results of this type were first obtained for the p Laplace equation by Krol' and Maz'ya in On the abscence of continuity and Hölder continuity of solutions of quasilinear elliptic equations near a nonregular boundary, Trans. Moscow Math. Soc. **26** (1972), 73 - 93. For existence they required $1 < p < n - 1$ and $\alpha \in (0, \pi)$, near enough π. Tolksdorf in On the Dirichlet problem for quasilinear elliptic equations in domains with conical boundary points, Comm. Partial Differential Equations **8** (1983), 773-817 extended these results (again for the p Laplace equation) to $\alpha \in (0, \pi)$ and $1 < p < \infty$. Also in Separable p-harmonic functions in a cone and related quasilinear equations on manifolds, JEMS **11** (2009), 1285-1305.
Porretta and Veron gave another proof of Tolksdorf's result and also considered a related Martin boundary problem. For existence and uniqueness of p harmonic functions u as in (14) in more general Lipschitz cones see Veron and Gkikas:
The spherical $p-$ harmonic eigenvalue problem in non smooth domains, Journal of Functional Analysis, in Press

For possible application to the smoothness problem involving μ in (13), we were mainly interested in finding $\lambda = \lambda(\pi)$ when $\alpha = \pi, p > n - 1$. To outline our efforts we assume for the moment in (14) that u is p harmonic in $K(\alpha)$. Then a natural way to attack this problem is to use the p Laplace equation, (14), and hopefully separation of variables to get a differential equation for ϕ. This was done by Krol' in On the behavior of the solutions of a quasilinear equation near null salient points of the boundary Proc. Steklov Inst. Math. **125** (1973) 130-136, who first obtained

$$0 = \frac{d}{d\theta}\left\{[\lambda^2\phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2}\,\phi'(\theta)\,(\sin\theta)^{n-2}\right\} +$$

$$\lambda[\lambda(p-1) + (n-p)][\lambda^2\phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2}\phi(\theta)(\sin\theta)^{n-2}$$

Second letting $\psi = \phi'/\phi$ in the above equation Krol' obtained,

$$\begin{aligned} 0 &= ((p-1)\psi^2 + \lambda^2)\,\psi' + \\ &(\lambda^2 + \psi^2)[(p-1)\,\psi^2 + (n-2)\cot\theta\,\psi + \lambda^2(p-1) + \lambda(n-p)] \end{aligned} \tag{15}$$

If $n = 2$ the cotangent term in the above DE goes out and variables can be seperated in (15) to get

$$\frac{\lambda d\psi}{\lambda^2 + \psi^2} - \frac{(\lambda - 1)\, d\psi}{\lambda^2 + \psi^2 + \lambda(2-p)/(p-1)} + d\theta = 0. \tag{16}$$

The boundary conditions imply that ϕ is decreasing on $(0, \alpha)$ so $\psi(\alpha) = -\infty, \psi(0) = 0$. Using this fact and integrating (16) it follows that

$$1 - \frac{\lambda - 1}{\sqrt{\lambda^2 + \lambda(2-p)/(p-1)}} = 2\alpha/\pi \tag{17}$$

which can be solved for λ in terms of α. if $\alpha = \pi, n = 2$, it follows from (17) that $\lambda(\pi) = 1 - 1/p$. Partly because of this result and partly because of what we thought we needed to handle the smoothness problem in (13) when $n > p > n - 1$, we conjectured what later turned out to be the following theorem:

Theorem 2

If $\mathcal{A} = \mathbb{D}f$, f as in (2), then $\lambda(\pi) = 1 - (n-1)/p$. In fact for $\alpha \in [3\pi/4, \pi)$

$$\lambda(\alpha) - 1 + (n-1)/p \approx (\pi - \alpha)^{\frac{p-n+1}{p-1}}$$

where $\approx$ means the rato of these functions is bounded above and below on the given interval by positive constants depending only on p, n and the structure constants for $\mathcal{A}$.

Remark Theorem 2 and Jerison - CNSXYZ arguments imply that if (13) holds in $\mathbb{R}^3$, when $1 < p < 3$ and $\mathcal{A} = \mathbb{D}f$, then Ω is $C^{k+2,\alpha}$.
Theorem 2 is used to handle the case when $2 < p < 3$.
In $\mathbb{R}^n$, $n > 3$, we can only get smoothness when $1 < p < 2$ (as in CNSXYZ for p harmonic functions). Admittedly at one time I thought Theorem 2 would imply this conclusion in $\mathbb{R}^n$ when $n-1 < p < n$.

We note that if $p \leq n-1$, then a slit has p capacity zero and it follows that there are no solutions to (3). In fact Krol' and Maz'ya in the paper mentioned earlier obtained for $1 < p < n-1$ that

$$\lambda(\alpha) = O\left((\pi-\alpha)^{(n-1-p)/(p-1)}\right) \text{ and } O\left(-\frac{1}{\log(\pi-\alpha)}\right)$$

for $p = n-1$ as $\alpha \to \pi$.

To outline our efforts in proving Theorem 2, we spent around 6 weeks trying to use the DE in (15) to determine λ when $\alpha = \pi$, $n \geq 3$, and u is p harmonic. From lower dimensional boundary Harnack inequalities of Lewis and Nyström we knew that

$$\lim_{\theta \to \pi} \psi(\theta)\,(\pi-\theta) = -\beta \tag{18}$$

where $\beta = \frac{1+p-n}{p-1}$. . Since ϕ has a relative maximum at $\theta = 0$, it also followed that $\psi(0) = 0$. Using these initial conditions, we first assumed for certain p, n that

$$\phi(\theta) = \cos(\theta/2)^{\beta}\, e^{g(\cos(\theta/2))}\,. \tag{19}$$

To test the validity of what was then a conjecture, we considered as a test case $n = 3, p = 5/2$, so $(\lambda(\pi) = 1/5)$ and in (19) put

$$g(\cos(\theta/2)) = \sum_{k=0}^{\infty} a_k (\cos(\theta/2))^{2k} \text{ where } a_k, k = 1, 2, \ldots \text{ are constants.} \tag{20}$$

From the series in (20) and our initial conditions one could see that the coeficients a_k could be computed recursively. Thanks to Maple and Mathematicia, we (Andy and Murat) could compute $a_1 - a_{10}$. Using the resulting partial sum for g, and then computing ψ we received strong evidence that $\lambda = 1/5$ in this test case. However we were never able to prove $\lambda = 1/5$. Finally we hit on using the following finess type proof of Theorem 2.

Ingredients of the Proof of Theorem 2 for p Harmonic Functions

To avoid confusion we shall sometimes write $u(\cdot, \alpha)$ for the function in (14). The first step in the proof is to show that $\lambda(\alpha)$ is strictly decreasing on $(0, \pi)$ so that

$$\lambda(\pi) = \lim_{\alpha \to \pi} \lambda(\alpha) \text{ exists and } \lambda(\alpha) \geq \lambda(\pi), \alpha \in (0, \pi]. \qquad (21)$$

Given (21) we use local L^p integrability of $|\nabla u|$ on the boundary of $K(\alpha)$ (as follows from references mentioned in the proof of Theorem 1 (a)) to get

$$\int_{\partial K(\alpha)\cap B(0,2)} |\nabla u|^p(x,\alpha)\, d\mathcal{H}^{n-1} \;\leq\; c(\alpha) \int_0^2 r^{(\lambda(\alpha)-1)p+(n-2)} dr < \infty.$$

Clearly this inequality and (21) imply

$$\lambda(\pi) = \lim_{\alpha \to \pi} \lambda(\alpha) \geq 1 - (n-1)/p. \tag{22}$$

To obtain an estimate for $\lambda(\pi)$ from above we shall need some notation. If $1 < p < n$, let

$$F(x) = c_p |x|^{(p-n)/(p-1)}.$$

Here $c_p = \frac{p-1}{n-p}\, \omega_n^{1/(1-p)}$ where $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \omega_n$.

Then as is easily checked,

$$\int_{\mathbb{R}^n} \langle |\nabla F|^{p-2} \nabla F, \nabla k \rangle dx = k(0), \text{ whenever } k \in C_0^\infty(\mathbb{R}^n) \,. \tag{23}$$

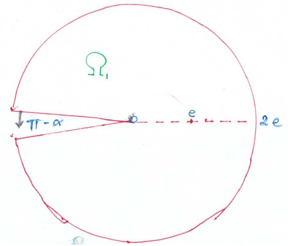
F is said to be a Fundamental solution to the p Laplace equation with pole at 0. Given a bounded connected open set Ω and $x_0 \in \Omega$ we say that G is Green's function for the p Laplace equation in Ω, with pole at x_0 provided

$$\begin{aligned}
&(a) \quad G \text{ is } p \text{ harmonic in } \Omega \setminus \{x_0\}, \\
&(b) \quad G \in W_0^{1,p}(\Omega \setminus B(x_0, \epsilon)), \text{ whenever } \epsilon > 0 \text{ and } B(x_0, 2\epsilon) \subset \Omega, \\
&(c) \quad F(x - x_0) = G(x) + \zeta(x), x \in \Omega, \text{ where } \zeta \text{ is bounded} \\
&\qquad \text{and Hölder continuous in a neighborhood of } x_0.
\end{aligned} \tag{24}$$

If also Ω is starlike Lipschitz we prove

$$\int_{\partial\Omega} |\nabla G(x)|^p \langle x - x_0, \nu \rangle \, dx = \frac{(n-p)}{p-1} \, \zeta(x_0) > 0 \qquad (25)$$

where ν is the outer unit normal to $\partial\Omega$. For $p = 2$ this inequality was proved by Jerison and Kenig in
Boundary value problems on Lipschitz domains, MAA Studies in Mathematics,Studies in Partial Differential Equations, **23** (1982),1-68.
Let G_1 denote the Green's function for
$\Omega_1 = B(0,2) \cap K(\alpha)$, $x_0 = (1, 0 \ldots, 0) = e$, and $0 < \pi - \alpha < \pi/10$. Also let G_2 denote the Green's function for $B(0,2)$ with pole at e. With this notation we proceed to the second step in our proof of Theorem 1 :



Lemma 1. For some $\tilde{c} \geq 1$, depending only on p, n it is true that

$$\int_{\partial K(\alpha)\cap B(0,2)} \langle \nu, x - e\rangle |\nabla G_1|^p d\mathcal{H}^{n-1} \geq \frac{n-p}{p-1}(\zeta_1 - \zeta_2)(e) \geq \tilde{c}^{-1} \quad (26)$$

where ν is the outer unit normal to $\partial K(\alpha)$ and ζ_1, ζ_2, are defined relative to G_1, G_2, as in (24) (c). To prove the key inequality in (26) we first use (25) for G_1, Ω_1. Next we note that $|\nabla G_1| \leq |\nabla G_2|$ on $\partial B(0,1) \cap \Omega_1$ as follows from the Hopf boundary maximum principle. Using this note and (25) for $G_2, B(0,2)$, we get the left inequality in (26). To prove the right hand inequality one needs to make estimates using the fact that $\zeta_1 - \zeta_2$ satisfies an elliptic PDE and the fact that a slit has positive capacity when $p > n - 1$. The idea to use a Rellich type inequality to make estimates as above we garnered from the paper of Venouziou and Verchota mentioned earlier.

In order to use Lemma 1 one first proves by way of a Hopf maximum type principle argument that t

$$|\nabla u(\cdot,\alpha)| \geq \bar{c}\,|\nabla G| \text{ on } \partial K(\alpha)\cap B(0,2) \tag{27}$$

and from a boundary Harnack inequality proved by Lewis and Nyström,, for $\mathcal{A}$ harmonic functions vanishing on lower dimensional Reifenberg flat sets (mentioned earlier) that

$$|\nabla u(\cdot,\alpha)| \leq c\,(\pi-\theta)^{(2-n)/(p-1)} \text{ on } \partial K(\alpha)\cap [B(0,2)\setminus B(0,1)] \tag{28}$$

where $\bar{c}, c$ depend only on p, n. Finally note that $\langle x-e,\nu\rangle = \sin(\pi-\alpha)$ on $\partial K(\alpha)\cap B(0,2)$. Using this note and (27), (28), in (26) we conclude for some $\breve{c}$ depending only on p, n that

$$\begin{aligned} c_p^{-1} &\leq \int_{\partial K(\alpha)\cap B(0,2)} \sin(\pi-\alpha)|\nabla G|^p\, d\mathcal{H}^{n-1} \\ &\leq \breve{c}\,(\int_0^2 r^{(\lambda_1(\alpha)-1)p+n-2} dr)\,(\pi-\alpha)^{\frac{p-n+1}{p-1}} \\ &\leq \frac{\breve{c}^2}{(\lambda_1(\alpha)-1)p+n-1}\,(\pi-\alpha)^{\frac{p-n+1}{p-1}}. \end{aligned} \tag{29}$$

where we have also used the fact that an element of surface area on $\partial K(\alpha)$ is of the form $[\sin(\pi-\alpha)]^{n-2} r^{n-2} dr$. From (29) and some arithmetic we conclude that

$$\lambda(\alpha) \leq 1-(n-1)/p + c^* (\pi-\alpha)^{\frac{p-n+1}{p-1}} \text{ as } \alpha \to \pi. \tag{30}$$

for some $c^* = c^*(p, n)$. Combining (30), (22), we get a weak version of Theorem 2 for p harmonic functions. $\square$

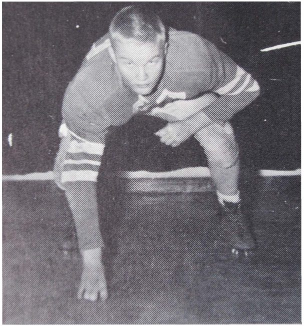
Query 1: Does Theorem 1 or 2 remain valid for a general $\mathcal{A}$ as in (1) ? Also in [AGHLV] we proved a Brunn Minkowski inequality for a general $\mathcal{A}$ as in (1) but could only handle the case of equality in BM if $\mathcal{A} = \nabla f$ and f is C^3 in $\mathbb{R}^n \setminus \{0\}$.

Coming soon?? $\lambda(\pi) = 1-(n-1)/p$ in Theorem 2 , $n \leq p < \infty$!

THANKS FOR LISTENING HOPE MY PRESENTATION WAS NOT TOO STORMY

DONALD

STORMY



STORMY'S SAVIOUR