Recent Results in Sparse Domination

Michael Lacey
Georgia Tech

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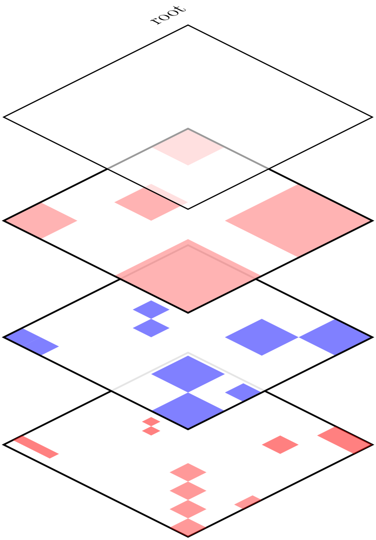
A Sparse Operator

A collection of cubes $\mathcal{S}$ is sparse if for each $S \in \mathcal{S}$, there is a an $E_S \subset S$, so that $|E_S| > \frac{1}{100}|S|$ and $\{E_S \ : \ S \in \mathcal{S}\}$ are disjoint.

$$\Lambda_{r,s}(f,g) = \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,r} \langle g \rangle_{S,s}.$$

$$\langle f \rangle_{S,r} = \Big[\frac{1}{|S|} \int_S |f| \, dx \Big]^{1/r}.$$



Definition

For sublinear operator T and $1 \le r, s < \infty$,

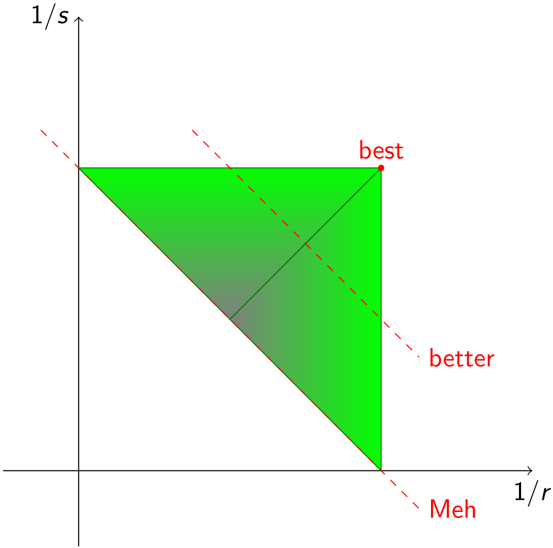
$$\|T \ : \ (r,s)\|$$

is the smallest constant C so that for all bounded compactly supported functions f, g

$$\underbrace{|\langle Tf, g\rangle|}_{\text{messy, complicated}} \le C \sup_{\Lambda} \underbrace{\Lambda_{r,s}(f,g)}_{\text{positive, localized}}$$

1. Definition only requires a bilinear form, not a linear operator.
2. The supremum over sparse forms is essentially obtained.
3. A $(1, r)$ bound implies weak-type, for any $r \ge 1$.
4. A (r, s) bound implies weighted inequalities: $r < p < s'$,

$$\|T \ : \ L^p(w) \mapsto L^p(w)\| \lesssim C(\|w\|_{A(p/r)}, \|w\|_{RH((s'/p)')})$$



1/s
best
better
1/r
Meh

The Sparse $T1$ Theorem

Theorem (L.-Mena)

Let T be a Calderón-Zygmund operator with kernel satisfying

$$|\nabla^{\alpha} K(x,y)| \le |x-y|^{-1-\alpha}, \qquad \alpha = 0, 1.$$

Assume for all cubes Q we have $\int_Q |T\mathbf{1}_Q| + |T^*\mathbf{1}_Q| \, dx \lesssim |Q|$. Then

$$\|T : (1,1)\| < \infty$$

- Many people contributed to this: Lerner, Conde-Rey, Hytönen, Volberg, Petermichl, Frey, Bernicot, di Plinio, Ou, Culiuc,.....
- This implies virtually all the standard mapping properties of T, with sharp constants (A_2 Theorem)
- Missing in this formulation: H^1/BMO type estimates.

Why is this (1,1) sparse bound true?

If f is supported on cube Q, then Tf is typically no more than $\langle f \rangle_Q$.

$$\| T \ : \ L^1_{loc} \mapsto L^\infty \| \text{``} < \infty \text{''}$$

Bilinear Hilbert Transform

$$BHT(f_1, f_2, f_3) = \int\int f_1(x-y) f_2(x-2y) f_3(x) \frac{dy}{y} dx$$

Theorem (Culiuc, di Plinio, Ou)

For admissible (p_1, p_2, p_3)

$$\|BHT \ : \ (p_1, p_2, p_3)\| < \infty.$$

For instance $(2, 2, 1)$ *is at the boundary of admissible.*

Sparse bounds have been proved for a wide variety of operators. Virtually the entire A_p literature has been completely rewritten in the last three years.

Along the way, bounds have been extended, simplified, and quantified.

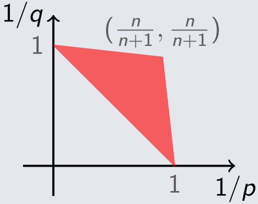
Littman/Strichartz Inequality

$$A_t f(x) = \int_{|y|=t} f(x-y)\, d\sigma(t)$$

Theorem (Littman (1971), Strichartz (1971))

For $(1/p, 1/q)$ are in the L^p improving triangle below,

$$\langle A_1 f, g\rangle \lesssim \|f\|_p \|g\|_q$$



Small Improvement, Inside the Triangle

$$\langle (A_1 - A_1 \circ \tau_y) f, g \rangle \lesssim |y|^{\delta_{p,q}} \|f\|_p \|g\|_q$$

Combine this with the Calderón-Zygmund-Christ method to deduce sparse bounds.

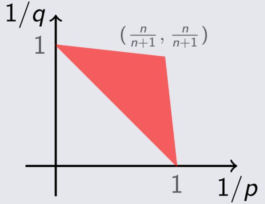
Lacunary Spherical Maximal Function

$$M_{\text{lac}} f(x) = \sup_{j \in \mathbb{Z}} \int_{\mathbb{S}^{n-1}} f(x - 2^j y)\, \sigma(dy)$$

Theorem (L.)

For $(1/p, 1/q)$ are in the L^p improving triangle below,

$$\|M_{\text{lac}} : (p, q)\| < \infty$$



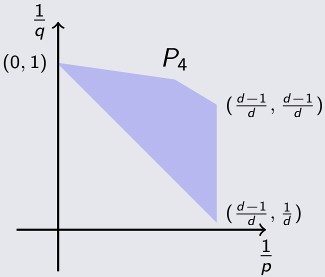
Stein Maximal Function

$$\tilde{M}f = \sup_{1 \le t \le 2} A_t f$$

Theorem (Schlag and Sogge)

For $(1/p, 1/q)$ are in the L^p improving triangle below,

$$\|\tilde{M} : (p,q)\| < \infty$$



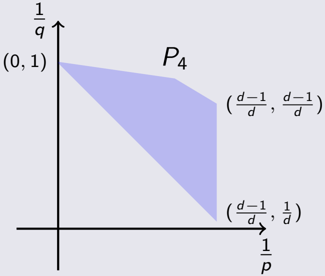
Stein Maximal Function

$$M_{\text{full}} f = \sup_{t>0} A_t f$$

Theorem

For $(1/p, 1/q)$ are in the L^p improving triangle below,

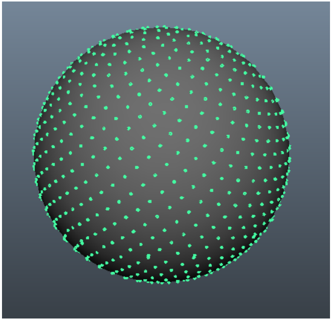
$$\|M_{\text{full}} \ : \ (p,q)\| < \infty$$



Discrete Spherical Averages

$$A_\lambda f(x) = \frac{1}{|\mathbb{Z}^d \cap \mathbb{S}_\lambda|} \sum_{n \,:\, |n|^2 = \lambda^2} f(x-n)$$

$$|\mathbb{Z}^d \cap \mathbb{S}_\lambda| \simeq \lambda^{d-2}, \qquad d \geq 5.$$



1. Started with Bourgain, and averages along square integers: $\frac{1}{N}\sum_{n=1}^{N} f(x-n^2)$
2. Discrete implies Continuous, but the two cases are dramatically different.
3. Entails Hardy-Littlewood method, and sometimes some serious number theory.
4. Many new difficulties, and fine distinctions with the continuous case.
5. Deep recent developments, including work of Bourgain, Mirek, Krause and Stein.

Magyar Stein Wainger Theorem (2002)

Theorem

For dimensions $d \geq 5$,

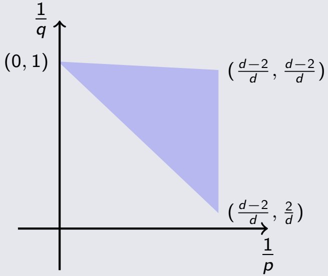
$$\|\sup_\lambda A_\lambda f\|_p \lesssim \|f\|_p, \qquad \frac{d}{d-2} < p < \infty$$

1. Compare to $\frac{d}{d-1}$ in the continuous case.
2. The case of 2, 3, 4 dimensions are excluded here, due to irregularities on the number of lattice points in these dimensions.

Theorem (Kesler, 2018)

For $(1/p, 1/q)$ *are in the triangle below,*

$$\|\sup_{\lambda} A_\lambda f \ : \ (p,q)\| < \infty$$



1. The sparse bound implies the ℓ^p-improving inequality, which is a result w/o precedent in the subject.

2. ℓ^p-improving does NOT imply the sparse bound. The 'Holder continuity' gain fails in the discrete setting, and there is no replacement for it.

3. Proof heavily expolits the representation of the multiplier from Magyar, Stein, Wainger.

4. The sparse bound implies a very rich set of weigthed and vector valued consequences, which are entirely new in this subject.

ℓ^p-improving in the fixed radius case

Theorem (Kesler-L (2018))

1. $\|A_\lambda f\|_{\ell^{p'}} \lesssim \lambda^{d(1-2/p)} \|f\|_p, \qquad \frac{d}{d-2} < p < 2$

2. $\|A_\lambda f\|_{\ell^{p'}} \lesssim C_{\omega(\lambda^2)} \lambda^{d(1-2/p)} \|f\|_p, \qquad \frac{d+1}{d-1} < p \leq \frac{d}{d-2}$ *where* $\omega(\lambda^2) =$ *number of distinct prime factors of* λ^2.

3. *If for all* $\epsilon > 0$, *all* λ, $\|A_\lambda f\|_{\ell^{p'}} \lesssim \lambda^{\epsilon + d(1-2/p)} \|f\|_p$, *then* $p \geq \frac{d+1}{d-1}$.

1. The sufficient proof uses
    1. Magyar's very fine analysis of the 'minor arcs.'
    2. Andre Weil's estimates for Kloosterman sums.
    3. A result of Bourgain on average values of Ramanjuan sums.

2. The necessary direction uses a subtle 'self-improving' aspect of the sufficient direction.

3. We do not know what the counterexample looks like!
   We just know that it exists.

4. These results hold in dimension $d = 4$, if λ^2 is odd.

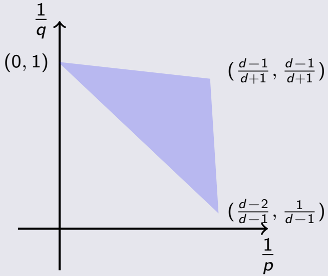
1. The theory of the discrete lacunary spherical maximal operator is rather different than the continuous case.
2. Due to an example of Zienkiewicz, there are lacunary radii λ_k for which $\sup_k A_{\lambda_k} f$ is *unbounded* for $1 < p < \frac{d}{d-1}$.
3. On the other hand, we should expect results for $A_{\mathrm{lac}} f = \sup_k A_{p^{k/2}} f$, for prime p, since $\omega(p^k) = 1$, for all primes p and integers k.
4. More evidence that the ℓ^p-improving and sparse bounds decouple in the discrete setting.

Sparse bounds of the discrete lacunary spherical maximal function

Theorem (Kesler-L, 2018)

For $(1/p, 1/q)$ *are in the triangle below,*

$$\|\sup_{\lambda} A_{\mathrm{lac}} f \ : \ (p,q)\| < \infty$$



Number Theory

$$A_\lambda f = C_\lambda f + R_\lambda f,$$

$$C_\lambda f = \sum_{1 \le \lambda \le q} \sum_{a=1}^{q} e(-\lambda^2 a/q) C_\lambda^{a/q} f,$$

$$c_\lambda^{a/q}(\xi) := \sum_{\ell \in \mathbb{Z}^d} G(a/q, \ell) \Phi_q(\xi - \ell/q) \widehat{d\sigma_\lambda}(\xi - \ell/q)$$

$$G(a/q, \ell) = q^{-d} \sum_{n \in \mathbb{Z}_q^d} e(|n|^2 a/q + n \cdot \ell/q).$$

$$K(\lambda, \ell, q) = \sum_{a=1}^{q} e(-\lambda^2 a/q) G(a/q, \ell)$$

Theorem (Magyar)

$$\|R_\lambda\|_{2\to 2} \lesssim \lambda^{-\frac{d-3}{2}}$$

Theorem (Weil)

$$|K(\lambda, \ell, q)| \lesssim q^{-\frac{d-1}{2}} \sqrt{(\lambda^2, q_{\mathrm{odd}}) q_{\mathrm{even}}}$$

$$\mathrm{c}_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{2\pi i n a/q}$$

Theorem (Bourgain)

For $n \neq 0$

$$\sum_{q=1}^{Q} |\mathrm{c}_q(n)| \lesssim Q^{1+\epsilon}$$

Alan McIntosh