The method of energy channels

Carlos Kenig
University of Chicago

Since the 1970's there has been a widely held belief that "coherent structures" describe the long term asymptotic behavior of solutions to nonlinear dispersive equations. This has come to be known as the "soliton resolution conjecture".

Roughly speaking, this means that asymptotically in time, the evolution of solutions of "most" nonlinear dispersive equations decouples as a sum of "traveling wave solutions" (modulated) and a free radiation term (a "dispersive" term solving the associated linear equation).

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This is a beautiful, remarkable claim, which shows a "simplification" in the complicated long-time behavior of solutions.

This "conjecture" has been one of the "holy grails" in the subject. Until recently, proved only in perturbative regimes, or for a few integrable equations (KdV, mKdV, cubic NLS on the line).

The mechanism for relaxation to a "coherent structure", which has been observed numerically and experimentally is the radiation of excess energy to spatial infinity.

This appears in such diverse settings as the dynamics of gas bubbles in a compressible fluid and in the formation of black holes in gravitational collapse.

In a series of works with Duyckaerts and Merle we have found a way to quantify the ejection of energy, for nonlinear wave equations, through a method that we call "energy channels". This has allowed us to make significant progress on "soliton resolution" (sometimes also with other collaborators).

This was accomplished as a consequence of some elementary inequalities ("outer energy lower bounds") for the linear wave equation.

The first one is:

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Let v be a radial solution of the linear wave equation in 3 space dimensions. Then, $\forall t \geq 0$ or $\forall t \leq 0$

$$(1) \quad \int_{|x| > |t| + r_0} |\nabla_{x,t} v|^2 \, dx \geq \frac{1}{2} \int_{r_0}^{\infty} \left[\partial_r (r v_0)^2 + r^2 v_1^2 \right] dx$$

where (v_0, v_1) is the initial data of v. (DKM 09)

In the nonradial case, we have, for v a solution of (LW) in $\mathbb{R}^N$, $N=3,5,7,\dots$

$\forall t \geq 0$ or $\forall t \leq 0$ [DKM 11]

$$(2) \quad \int_{|x|\geq|t|} |\nabla_{x,t} v|^2 \geq \frac{1}{2}\int |\nabla v_0|^2 + v_1^2$$

When $r_0=0$ in (1), we have the radial case of (2).

Nonlinear applications of the "energy channels" created by (1), (2).

We start with the energy critical nonlinear wave equation. (3d)

$$(NLW)\begin{cases} \partial_t^2 u - \Delta u - u^5 = 0 & (|u|^{4/N-2} u) \\ & (x,t) \in \mathbb{R}^3 \times I \\ u|_{t=0} = u_0 \in \dot{H}^1, \ \partial_t u|_{t=0} = u_1 \in L^2 . \end{cases}$$

Here I is a time interval, $0 \in I$.

For (NLW), small data yield global in time solutions, which scatter.

For large data we have solutions $u \in C(I; \dot{H}^1 \times L^2)$ with a maximal interval of existence $I = (T_-(u), T_+(u))$ with $(N=3)$ $u \in L^5_{I'} L^{10}_x$ $\forall I' \subset\subset I$.

The energy norm is critical since $\forall \lambda > 0$ $u_\lambda(x,t) = \lambda^{-1/2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$ $(\lambda^{-N-2/2})$ is also a solution, and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} =$

$= \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$.

The equation is Focusing, the conserved energy is

$$E(u_0, u_1) = \frac{1}{2}\int |\nabla u_0|^2 + u_1^2$$
$$- \frac{1}{6}\int u_0^6 \quad \left(\frac{|u_0|^{2N/N-2}}{2N/N-2}\right).$$

It is easy to construct solutions which blow-up in finite time by the ODE and finite speed of propagation.

These solutions have

$$\lim_{t \uparrow T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = \infty$$

(type I blow-up).

There are also type II blow-up solutions: $T_+ < \infty$ but

$$\sup_{0 < t < T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty.$$

The breakdown occurs by concentration, a typical feature of energy critical. ($N=3$ Krieger-Schlag-Tataru, $N=4$ Hillairet-Raphaël, $N=5,6$ Jendrej)

For this equation, "soliton resolution" is expected for bounded solutions:

$$\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty, \quad 0 < T_+ \le \infty.$$

Examples of such solutions, with $T_+ = \infty$, are scattering solutions:

$\exists (u_0^+, u_1^+) \in \dot{H}^1 \times L^2$ s.t.

$$\lim_{t \uparrow \infty} \|(\vec{u}(t) - \vec{S}_L(t)(u_0^+, u_1^+)\|_{\dot{H}^1 \times L^2} = 0$$

For instance, for (u_0, u_1) small we have a scattering solution.

Other examples of bounded solutions, $T_+ = \infty$ are stationary solutions:

$Q \not\equiv 0$, $Q \in \dot{H}^1$ s.t.

$$\Delta Q + Q^5 = 0 \quad (|Q|^{4/N-2} Q).$$

We say $Q \in \Sigma$.

Stationary solutions do not scatter (they don't disperse).

An example is

$$W(x) = \left({}^{1}/_{1+|x|^2/3} \right)^{1/2}$$

$$\left({}^{1}/_{1+|x|^2/N(N-2)} \right)^{N-2/2}$$

W is up to sign, scaling is the only radial static solution. Up to translation and scaling it is the only non-negative static solution. W is a nonlinear ground state. Up to translation, scaling and sign it is the unique minimiser in Σ, of the energy.

Σ is complicated and has not been classified.

Other bounded, non-scattering solutions: the traveling waves.
They are Lorentz transforms of $Q \in \Sigma$. Let $\vec{\ell} \in \mathbb{R}^3$, $|\vec{\ell}|<1$. Then,

$$Q_{\vec{\ell}}(x,t) = Q_{\vec{\ell}}(x-\vec{\ell}t, 0) =$$

$$Q\left(\left[\frac{-t}{(1-|\vec{\ell}|^2)^{1/2}} + \frac{1}{|\vec{\ell}|^2}\left(\frac{1}{(1-|\vec{\ell}|^2)^{1/2}} - 1\right)\vec{\ell}.x\right]\vec{\ell} + x\right)$$

is a traveling wave solution. It does not scatter.
DKM14: These are the only traveling waves.

Radial Case: Only static solutions $\pm W_\lambda$, no traveling waves.

Solition resolution for (NLW), radial case.

Proved for $N=3$, DKM12 for radial solutions bdd in norm, for well-chosen sequences of time, general time sequences DKM13, who gave a classification.

First (and main) step in DKM13:

Dynamical characterization of W, through energy channels.

If u is a radial solution of (NLW), $N=3$, which exists for all times and is not $0, \pm W_\lambda$, then $\exists R>0, \eta>0$ s.t.:
$\forall t \geq 0$ or $\forall t \leq 0$

$$(*) \quad \int_{|x|>|t|+R} |\nabla_{x,t} u|^2 \, dx \geq \eta$$

The outer energy lower bound (1) easily gives $(*)$ for radial solutions of (LW), $N=3$.
The passage to nonlinear solutions uses (1) and "elliptic arguments" of iterative type.

(*) is the key tool for:

Theorem (DKM13): Let u be a radial solution of (NLW). Then one of the following holds:

i) $T_+ < \infty$ and $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1_x L^2} = \infty$

(type I blow-up)

ii) $T_+ < \infty$ and $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1_x L^2} < \infty$

(type II blow-up) and

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$\exists J \geq 1$, $i_j \in \{\pm 1\}$, $\lambda_j(t) > 0$,
For $1 \leq j \leq J$, with
$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll (T_+ - t)$$
such that
$$\vec{u}(t) - \vec{v}(t) = \sum_{j=1}^{J} i_j \left(W_{\lambda_j(t)}, 0\right) + o_{\dot{H}^1 \times L^2}(1),$$
where $\vec{u}(t) \xrightarrow[t \to T_+]{} (v_0, v_1)$ and v is the sol'n of (NLW) with $\vec{v}(T_+) = (v_0, v_1)$ (the regular part of u) which verifies
$$\operatorname{supp}\left(\vec{u}(t) - \vec{v}(t)\right) \subset \{|x| < T_+ - t\}.$$

iii) $T_+ = \infty$. Then $\exists\, v_L$ a soltn (21)
of (LW) and $J \geq 0$, for $0 \leq j \leq J$, $i_j \in \{\pm 1\}$
$\lambda_j(t) > 0$ with

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll t$$

and

$$\vec{u}(t) - \vec{v}_L(t) = \sum_{j=1}^{J} i_j \left(W_{\lambda_j(t)}, 0 \right) + o_{\dot{H}^1 \times L^2}(1).$$

To see the "energy channels" method
at work, we'll sketch a proof of ii),
$T_+ = 1$.

Take $t_n \to 1$. We break up $\vec{u}(t_n) - \vec{v}(t_n)$ (which is supported in $|x| < (1-t_n)$) into a sum of "orthogonal" nonlinear "blocks" (technically, nonlinear profiles U_j associated to a Bahouri-Gérard profile decomposition, plus an error w_n, which is a linear solution tending to 0 in the weaker "dispersive" norm $L^5_t L^{10}_x$ $(N=3)$). This can be done through an approximation Thm of Bahouri-Gérard.

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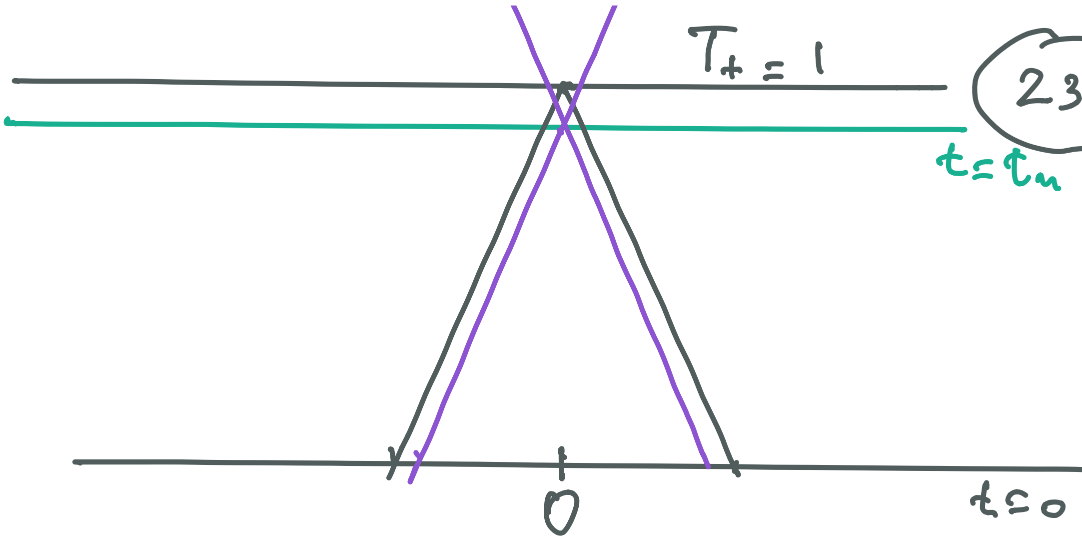

If a "block" U_j is not $\pm W_{\lambda_j}$, by $(*)$ it will send energy outside the light cone at $t=1$ (case $t \geq 0$ in $(*)$) a contradic. to the support property of $\vec{u}-\vec{v}$, or arbitrarily close to the boundary of the light cone at $t=0$, a contradic. to $\vec{u}(0)-\vec{v}(0) \in \dot{H}^1 \times L^2$.

Finally one uses (2) to show the dispersive errors $\vec{w}_n$ tend to 0 in energy norm, by a similar argument.

The earlier argument of DKM12, for well-chosen time sequences is also very useful. Say again $T_+ = 1$, then DKM12 showed, using (1) that for radial type II blow-up, $N=3$,

"no self-similar blow-up is possible":

$\forall\ 0<\lambda<1$

$$(**)\quad \lim_{t\uparrow 1} \int_{\lambda(1-t)<|x|<(1-t)} |\nabla_{x,t} u(x,t)|^2 dx = 0$$

Later, a different proof of $\binom{*}{*}$ was give (Côte-K-Lawrie-Schlag 14), which applied for radial solutions, all N. This was through an adaptation of the classical argument for equivariant wave maps, due to Christodoulou, Shatah, Tahvildar-Zadeh, (early 90's).

Combining $\binom{*}{*}$ with "virial identities" DKM 12 showed

$$\begin{pmatrix}*\\ *\\ *\end{pmatrix} \quad \lim_{t \uparrow 1} \int_t^1 \int_{|x|<1-t} (\partial_t u(x,s))^2 \, dx \, ds = 0.$$

From this, one shows by a Tauberian argument, that for well-chosen $t_n \to 1$, all nonlinear blocks are time independent and thus $\pm W_{\lambda_j}$ and

$$\int [\partial_t u(t_n) - \partial_t v(t_n)]^2 dx \longrightarrow 0.$$

Finally, using (2), the dispersive errors tend to 0 in energy norm, by the "energy channels" argument.

Moving Forward:

Fact 1: (Côte-K-Schlag 13)
(1) and (2) (outer energy ineq.) fail for all even N, radial solutions of (NLW).
However (2) holds $N=4,8,\ldots$ for $(v_0,v_1)=(v_0,0)$, for $N=6,10,$ for $(v_0,v_1)=(0,v_1)$.

Fact 2: (K-Lawrie-Liu-Schlag 14)
An analog of (1) holds for all odd N, radial case, but with exceptional set of functions of finite, but increasing with N dimension

Fact 3: (DKM 12) The analog of (1) fails in all dims, non-radial case.

Using Fact 2, C. Rodriguez 14 extended DKM12, for well chosen time sequences, all odd dims. Using Fact 1 and that for $N=4$ good data are $(v_0, 0)$, $\partial_t w_n \to 0$, Côte-K-Lawrie-Schlag 14 extended DKM12 to $N=4$. Côte-K-Lawrie-Schlag 13, Côte 14 did same for co-rotational wave maps into sphere. Jia-K 15 introduced new virial method, without (1), (2), to treat (NLW) $N=6$, equiv. wave maps into sphere, radial Yang-Mills $N=4$, for well chosen seq. of times.

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Non-radial case of bdd. solutions of (NLW).

New difficulties: The set of travelingwaves $Q_{\vec{\ell}}$ is very large, far from understood.

The outer energy lower bounds (1), (2) for (LW) fail in nonradial case.

Thus: An approach based on dynamical characterization of traveling waves seems doomed to failure.

Work of DKM 16, DKJM 16

We consider bded solutions of (NLW) ($\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < \infty$) in $\mathbb{R}^N$, $N=3,4,5,6$. If $T_+<\infty$, we have type II blow-up. The set S, of singular points is the set where the solution concentrates energy at the blow-up time. DKM11: $S \neq \phi$, finite. Moreover $\vec{u}(t) \xrightarrow[t\to T_+]{} (v_0, v_1)$ and if v solves (NLW) $\vec{v}(T_+) = (v_0, v_1)$,

$$\operatorname{supp}\left[\vec{u}(t) - \vec{v}(t)\right] \subset \bigcup_{k=1}^{M} \{(x,t) : |x - x_k| < |T_+ - t|\}, \quad S = \{x_1, \qquad , x_M\}.$$

In radial case $S = \{0\}$.

Theorem (DJKM 16)

i) $T_+ < \infty$. Fix $x_0 \in S$. Then $\exists\, J \geq 1$, $r_0 > 0$, time sequences $t_n \uparrow T_+$ (well-chosen), scales $0 < \lambda_n^j << T_+ - t_n$, positions $c_n^j \in B_{\beta(T_+ - t_n)}(x_0)$, $\beta \in [0,1)$ with $\vec{l}_j = \lim_n \dfrac{c_n^j}{T_+ - t_n}$ well-defined, and traveling waves $Q^j_{\vec{l}_j}$, $1 \leq j \leq J$, s.t. for $x \in B_{r_0}(x_0)$, we have:

$$\vec{u}(t_n) - \vec{v}(t_n) =$$

$$\sum_{j=1}^{J} \left((\lambda_n^j)^{-(N-2)/2} Q^j_{\vec{l}_j}\left(\frac{x - c_n^j}{\lambda_n^j}, 0\right), \right.$$

$$\left. (\lambda_n^j)^{-N/2} \partial_t Q^j_{\vec{l}_j}\left(\frac{x-c_n^j}{\lambda_n^j}, 0\right) \right) +$$

$$+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty.$$

Moreover,

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|c_n^j - c_n^{j'}|}{\lambda_n^j} \xrightarrow[n]{} \infty \quad j \neq j'.$$

(33)

ii) $T_+ = \infty$. Then (DKM16)

$\exists! \, v_L$ solving (LW) (radiation) s.t.

$$\lim_{t\to\infty} \int_{|x|>t-A} |\nabla_{x,t}(u - v_L)(x,t)|^2 dx = 0 \quad \forall A$$

Moreover $\exists \, t_n \to \infty$ (well-chosen), $\exists J \geq 0$, $\lambda_n^j > 0$, $c_n^j \in \mathbb{R}^N$, $\in B_{\beta t_n}(0)$, $\beta \in [0,1)$,

$\lim_n \frac{c_n^j}{t_n} = \vec{\ell}_j$, traveling waves $Q^j_{\vec{\ell}_j}$ s.t.

$$\vec{u}(t_n) - \vec{v}_L(t_n) = \sum_{j=1}^{J} \Big((\lambda_n^j)^{-N-2/2} Q^j_{\vec{\ell}_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big),$$

$$(\lambda_n^j)^{-N/2} \partial_t Q^j_{\ell_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big) \Big) +$$

$$o_{\dot{H}^1 \times L^2}(1),$$

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|c_n^j - c_n^{j'}|}{\lambda_n^j} \xrightarrow[n]{} \infty \quad j \neq j'.$$

Remark: The proof of i) ($T_+<\infty$) with error $(\varepsilon^0_n, \varepsilon^1_n)$ verifying the weaker conclusion $\vec{S}_L(\varepsilon^0_n, \varepsilon^1_n) \xrightarrow[n]{} 0$ in $L^5_t L^{10}_x$ ($N=3$) ("dispersive" norm) is due to Hao Jia 15.

The extraction of the radiation term in ii) is delicate (DKM16). We have $\vec{v}_L = \vec{S}_L(v_0, v_1)$, with

$$\vec{S}_L(-t)(\vec{u}(t)) \xrightarrow[t\to\infty]{} (v_0, v_1).$$

The starting point of the proof is the following Morawetz estimate (say when $T_- = 0$, $0 \in S$):

$$\int_{t_1}^{t_2} \int_{|x|<t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1\right) \frac{u}{t} \right]^2 dx \frac{dt}{t} \leq$$

$$\leq C(u) \left[\log t_2/t_1 \right]^{1/2}.$$

From this and Tauberian arguments we show that $\exists$ (many) $t_n \downarrow 0$, with

$$\sup_{0<\tau<t_m/16} \frac{1}{\tau}\int_{|t_m-t|\le\tau}\int_{|x|<ct}\left[\partial_t u+\frac{x}{t}\cdot\nabla u+\left(\frac{N}{2}-1\right)\frac{u}{t}\right]^2 dx\,dt$$

$$\xrightarrow[m]{} 0 .$$

From this a preliminary decomposition with a dispersive error is obtained. To show the error goes to 0 in energy, we use a new "energy channels" argument, valid in non-radial case, all dims.

Here is the 3d version 37

<u>Prop</u> (DJKM16) Let $(\varepsilon^0_m, \varepsilon^1_m)$ be a bded seq. in $\dot H^1 \times L^2$ s.t. $\forall \lambda \in (0,1)$ we have

$$\|\varepsilon^0_m\|_{L^6} + \|(\varepsilon^0_m, \varepsilon^1_m)\|_{\dot H^1 \times L^2(B_\lambda \cap B_1^c)} +$$

$$+ \left\|\nabla \varepsilon^0_m - \frac{x}{|x|}\nabla \varepsilon^0_m\right\|_{L^2} +$$

$$+ \|\varepsilon^1_m + \partial_r \varepsilon^0_m\|_{L^2} \xrightarrow[m]{} 0. \text{ Then}$$

if $\inf \|(\varepsilon^0_m, \varepsilon^1_m)\|_{\dot H^1 \times L^2} > 0$, $\vec{v}_m = \vec{S}_L(t)(\varepsilon^0_m, \varepsilon^1_m)$, $\forall \eta_0 \in (0,1)$, n large

$$\int_{|x| \geq t} |\nabla_{x,t} v_m(t)|^2 dx \geq \eta_0/2 \, \|(\varepsilon^0_m, \varepsilon^1_m)\|^2_{\dot H^1 \times L^2}$$

$\forall t \geq 0$.

These ideas are very robust. In DJKM 17, we have begun our study of energy critical wave maps into S^2, without symmetry. For this we expect soliton resolution for all t.

We have shown, in the finite time blow-up case that if the energy is only slightly higher than the one of the degree one co-rotational harmonic map, the wave map consists essentially of two parts:

A regular part outside the light cone $|x| > T_+ - t$, and a traveling wave (built by Lorentz transform of a modulated co-rotational degree 1 harmonic map), with small velocity $\vec{\ell}$, that concentrates in a small region (relative to the size of the light cone), near the point $\vec{\ell}(T_+ - t)$.

The strategy of proof uses a preliminary decomposition, valid only strictly inside the cone, due to R. Grinis, and an "energy channels" argument.

Prop (DJKM17): Fix $\beta \in (0,1)$. 41

$\exists\, \delta = \delta(\beta), \varepsilon_0 = \varepsilon_0(\beta)$ small s.t. if u is a classical wave map, $\mathcal{E}(u) < \varepsilon_0$, with

$$\|(u_0,u_1)\|_{\dot H^1 \times L^2(B_{1+\delta}^c \cup B_{1-\delta})} + \left\|\nabla u - \frac{x}{|x|}\cdot \nabla u\right\|_{L^2} + \|u_1 + \partial_r u_0\|_{L^2} \le \delta \|(u_0,u_1)\|_{\dot H^1 \times L^2},$$

then $\forall t \ge 0$

$$\int_{|x| > t+\beta} |\nabla_{x,t} u(x,t)|^2 \, dx \ge \beta \|(u_0,u_1)\|^2_{\dot H^1 \times L^2}$$

Extensions to the general case are in progress.

Concerning well-chosen time seq's

The methods of proof using monotonicity after time averaging cannot give more than a decomposition for (many) well-chosen sequences of time.

The difficulty is illustrated by the harmonic map heat flow. Here, the analog of our results is known, but the resolution for all times does not hold in full generality (Topping).

The main challenge is, using the decomposition in our results as a first step, to prove that the collision of two or more traveling waves produces dispersion.

This is the subject of intensive investigation. Note that in the radial 3d case of (NLW), this is a consequence of the dynamical characterization (*) of W, explained earlier.

Thank you for your

attention