The Two Hyperplane Conjecture

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Outline

- The two hyperplane conjecture
- Whence it came (level sets)
- Where it may lead
- Quantitative connectivity: Isoperimetric, Poincaré and Harnack inequalities.

Isoperimetric set E relative to μ

$$P_\mu(E) \le P_\mu(F) \text{ for all } F, \quad \mu(F) = \mu(E).$$

Perimeter of E relative to a measure μ on $\mathbb{R}^n$

$$P_\mu(E) := \liminf_{\delta \searrow 0} \frac{\mu(E_\delta) - \mu(E)}{\delta} \quad (E_\delta = \delta\text{-nbd of } E).$$

Example: $\mu = 1_\Omega dx$, Ω **open, convex:**

$$P_\mu(E) = H_{n-1}(\Omega \cap \partial E) \quad (E \text{ open}).$$

Conjecture 1. There is $b(n) > 0$ such that if $\Omega \subset \mathbb{R}^n$ is convex, symmetric ($-\Omega = \Omega$), $E \subset \Omega$ is isoperimetric, $|E| = |\Omega|/2$, then there is a half space H such that

$$H \cap \Omega \subset E, \quad (-H \cap \Omega) \subset \Omega \setminus E, \quad |H \cap \Omega| \geq b(n)|\Omega|.$$

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Conjecture 1^* **(Two hyperplane conjecture)**
$b(n) \geq b^* > 0$ an absolute constant.

Conjecture 2 (qualitative form). If $\Omega \subset \mathbb{R}^n$ is convex, $E \subset \Omega$ is isoperimetric, $0 < |E| < |\Omega|$, then

$$\mathrm{hull}(E) \neq \Omega$$

Open question, even in $\mathbb{R}^3$.

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First enemy of both conjectures:

$$E = \Omega \setminus B \quad (B \ \text{ a ball}).$$

First serious enemy: the Simons cone

$$S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| = |y|\} \subset \mathbb{R}^{2n}, \quad 2n \geq 8.$$

S is area-miniminizing for fixed boundary conditions in any Ω.

$$S_1 = S \cap B_1, \quad \Omega := \mathrm{hull}(S_1),$$

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E is not stable for the isoperimetric problem.

This is a slightly sharpened version of a theorem of Sternberg and Zumbrun from 1990s.

Log-concave measures on $\mathbb{R}^n$

$$\mu = e^{-V} dx, \quad V \text{ is convex.}$$

Ω convex is achieved in the limit:

$$\mu = 1_\Omega dx; \quad V(x) = 0,\ x \in \Omega, \quad V(x) = \infty,\ x \in \mathbb{R}^n \setminus \Omega.$$

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KLS Hyperplane Conjecture. There is an absolute constant $c^* > 0$ such that if μ is log-concave on $\mathbb{R}^n$ and E is isoperimetric with $\mu(E) = \mu(\mathbb{R}^n)/2$, then there is a half space H for which

$$P_\mu(E) \geq c^* P_\mu(H), \quad \mu(H) = \mu(E).$$

Proposition (E. Milman) "Two implies One". Suppose that μ is log-concave, E is an isoperimetric set with $\mu(E) = \mu(\mathbb{R}^n)/2$, and there are half spaces H_i such that

$$\mu(H_i) \geq b^* > 0, \quad H_1 \subset E, \quad H_2 \subset \mathbb{R}^n \setminus E.$$

Let H_0 be the translate of H_1 such that $\mu(H_0) = \mu(E)$. Then

$$P_\mu(E) \geq c^* P_\mu(H_0), \quad c^* = \frac{1}{4\log(1/b^*)}.$$

Conjecture 3. (Half space conjecture)
If $\nabla^2 V >> 0$ and $E \subset \mathbb{R}^n$ is isoperimetric for $\mu = e^{-V}dx$, with $\mu(E) = \mu(\mathbb{R}^n)/2$, then there are convex sets K_1 and K_2 such that

$$K_1 \subset E, \quad K_2 \subset \mathbb{R}^n \setminus E, \quad \mu(K_i) \geq c > 0$$

for an absolute constant c. Moreover, at least one of the two sets K_i can be taken to be a half space.

First Variation for $\mu = w\,dx$

$$H_\mu = (n-1)H - \mathbf{n} \cdot \nabla V, \quad w = e^{-V}.$$

Second Variation (stability) for $S = \partial E$

$$\int_S (|A|^2 + \nabla^2 V(\mathbf{n},\mathbf{n})) f^2 w\, d\sigma \;\le\; \int_S |\nabla_S f|^2 w\, d\sigma$$

$$\text{provided} \quad \int_S f\, w\, d\sigma = 0.$$

Symmetry Breaking

Proposition (variant of Sternberg-Zumbrun) If E is isoperimetric for $\mu = e^{-V}dx$, $\nabla^2 V >> 0$, and

$$V(-x) = V(x), \quad -E = E,$$

then E is not stable.

Proof: Take $f_j = \mathbf{e}_j \cdot \mathbf{n}$ (orthonormal basis $\mathbf{e}_j$).

$$\sum |\nabla f_j|^2 = |A|^2, \quad \sum f_j^2 = 1$$

Rediscovered by Rosales, Cañete, Bayle and Morgan in radial case.

Let $C(\mu)$ be the best constant in Poincaré's inequality

$$\int |f|^2 \, d\mu \leq C(\mu) \int |\nabla f|^2 \, d\mu \qquad \int f \, d\mu = 0. \qquad (*)$$

(KLS $\iff$ *linear* test functions suffice.)

When $\mu = 1_\Omega \, dx$, extremals u are Neumann eigenfunctions for $\lambda = 1/C(\mu)$:

$$\Delta u = -\lambda u \text{ in } \Omega, \quad \nu \cdot \nabla u = 0 \text{ on } \partial\Omega,$$

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Our version for today: If $\Omega \subset \mathbb{R}^n$ is convex, open, bounded, and $-\Omega = \Omega$, then each first Neumann eigenfunction is monotone:

$$e \cdot \nabla u > 0 \text{ in } \Omega \quad \text{(for some direction } e\text{)}.$$

Two axes of symmetry: J-, Nadirashvili 2000.

"lip domains", obtuse triangles: Atar-Burdzy 2004,

acute triangles: Judge-Mondal (preprint).

N. B. With monotonicity (and strict convexity) the level sets are topologically trivial, smooth graphs. Some extra hypothesis like $-\Omega = \Omega$ is needed to get monotonicity. Already for many acute triangles some level sets are disconnected.

Deformation approach to hot spots

Consider u_t, Ω_t, $0 \leq t \leq 1$, and

$$G = \{t \in [0,1] : e \cdot \nabla u_t(x) > 0 \text{ in } \Omega_t\}$$

$0 \in G$, G is open and closed $\implies G = [0,1]$.

Show G is closed by showing that the level sets

$$\{x \in \Omega_t : u_t(x) = c\}$$

are Lipschitz graphs with vertical direction e.

A priori Lipschitz bounds

Theorem (Bombieri, De Giorgi, Miranda) If $\varphi \in C^\infty(B_1)$ satisfies

$$\nabla \cdot \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) = 0$$

with $|\varphi| \leq M$, then there is $C = C(n, M)$ s. t.

$$|\nabla \varphi| \leq C \text{ in } B_{1/2}.$$

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Proof (De Silva, J-)

$$S_r := \{(x, \varphi(x)) : x \in B_r\}$$

We want to prove

$$\mathrm{dist}(S_1, S_1 + (0, \varepsilon)) \geq c\,\varepsilon$$

Step 1. The normal distance is $\geq c_1\varepsilon$ on a "good" set G of large measure because

$$\int_{B_{3/2}} |\nabla\varphi|\, d\sigma \leq CM^2.$$

Step 2. Define the normal variation $\psi(X)$ by

$$(\Delta_S + |A|^2)\psi = 0 \quad \text{on } S_2 \setminus G.$$

$\psi = 1$ on G, $\psi = 0$ on ∂S_2. Our goal

$$S(t) = \{X + t\psi(X)\nu : X \in S_1\}$$

cannnot touch $S_1 + (0, \varepsilon)$ for $0 \le t \le c_1\varepsilon$ for $X \in G$.

Step 3.

$$\Delta_S w \leq 0 \quad \text{(supersolution)}$$

Bombieri-Giusti Harnack inequality:

$$\inf_{S_1} w \geq c_2 \int_{S_1} w \, d\sigma \geq c_2 \sigma(G)$$

Hence,

$$\psi \geq c > 0 \quad \text{on } S_1.$$

Hence we have normal separation by $c\varepsilon$ on all of S_1.

Free boundary gradient bound

Thm (De Silva, J-) If u is positive, harmonic in

$$\{(x,y) : y > \varphi(x)\}, \quad |\nabla u| = 0 \text{ on } \{y = \varphi(x)\},$$

and $|\varphi| \leq M$ on $|x| < 2$, then

$$|\nabla \varphi| \leq C_M \quad \text{on } |x| < 1.$$

Proof: $y > \varphi(x)$ is an NTA domain and the

boundary Harnack inequality plays the role analogous to the Bombieri-Giusti intrinsic Harnack inequality.

Bombieri-Giusti Harnack is deduced via Moser type argument from De Georgi local isoperimetric inequality:

$$\min_k \int_{B_{\beta R}} |f - k|^{n/(n-1)} \, d\sigma \le C \int_{B_R} |\nabla_S f| \, d\sigma \, .$$

Proof is via blow up and compactness from qualitative (measure-theoretic) connectivity of area minimizing cones due to Almgren and De Giorgi.

Proposition Let $S \subset \mathbb{R}^n$ be an area minimizer, then S divides $\mathbb{R}^n$ into two NTA domains.

Lemma If $E = A_1 \cup A_2$, $|A_1 \cap A_2| = 0$, then for some $\beta > 0$,

$$P(A_i, E \cap B_r) \geq c \min_i (|A_i \cap B_{\beta r}|)^{(n-1)/n}$$

The lemma is proved by the method of Almgren-De Giorgi. Then methods of G. David and S. Semmes yield the proposition.

For isoperimetric surfaces in convex domains, the first important steps have been taken by Sternberg and Zumbrun, who showed that the stability implies an L^2 Poincaré inequality. They deduced that the isoperimetric sets and the interfaces are connected.

With G. David, we hope to prove a scale invariant Poincaré inequality *up to the boundary* of the convex domain. This should yield the regularity needed to perform the De Silva type argument, provided one can get started with the right global estimate.

Theorem (G. David, D.J.) If S is an area minimizing surface in $\mathbb{R}^n$, then the intrinsic distance on S is equivalent to Euclidean distance.

Main Lemma: Intrinsic balls of radius r have area $\geq cr^{n-1}$.

Calabi-Yau Conjecture
(proved by Colding-Minicozzi) The only complete, embedded minimal surface with finite topology in a half space in $\mathbb{R}^3$ is the plane.

Key lemma: The embedding is proper. This is a qualitative version of the statement that intrinsic distance is comparable to ambient distance.

Happy Birthday, Steve!

$f(y) := \tilde{d}(x, y)$, intrinsic distance on $S = \partial E$.

1. There is $r/2 < \rho < r$ such that
$$\text{mass}(S_\rho) \le \frac{1}{r}\sigma(\tilde{B}_r)$$
2. There is an integral current T, $\partial T = S_\rho$ and
$$\text{mass}(T) \le c_n(\text{mass}(S_\rho))^{(n-1)/(n-2)}.$$
3. If $\sigma(\tilde{B}_r) << r^{n-1}$, then
$$\sigma(\text{supp}(T)) << \sigma(\tilde{B}_r).$$

Isoperimetric ineq of De Giorgi implies

$$\min_a \int_{S\cap B_r} |f(y)-a|\, d\sigma \le Cr \int_{S\cap B_{Cr}} |\nabla_S f|\, dH^{n-1}$$

For all $z \in B_{r/2}(x)\cap S$ and all $y \in \tilde{B}_{r/2}(z)$,

$$|f(z)-a| - r/2 \ \le \ |f(y)-a|$$

$$|f(z)-a| - r/2 \le \frac{C}{r^{n-1}} \int_{\tilde{B}_{r/2}(z)} |f(y)-a|\, d\sigma$$

$$\le \ \frac{Cr}{r^{n-1}} \int_{B_{Cr}(x)} |\nabla_S f|\, d\sigma \le \ Cr.$$

In conclusion, for all $z \in B_{r/2}(x)$,

$$\tilde{d}(x,y) = |f(z)| = |(f(z) - a) - (f(x) - a)| \leq 2Cr.$$

as desired.