Finite point configurations and frame theory: connections and perspectives

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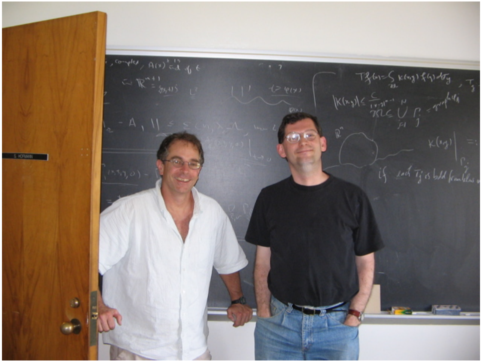
15 years ago



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- A wide variety of problem in mathematics and computer science fall in this category.

Fourier bases and tiling

- Let Ω be a domain in $\mathbb{R}^d$ or $\mathbb{F}_q^d$. When does $L^2(\Omega)$ possess an orthogonal basis (or a Riesz basis) of the form

$$\{e^{2\pi i x \cdot a}\}_{a \in A}$$

in Euclidean space, and an orthogonal basis of the form

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- A related question, as it turns out, is whether a given domain Ω in $\mathbb{R}^d$, or $\mathbb{F}_q^d$ tiles the whole space by translation, i.e whether there exists $T \subset \mathbb{R}^d$, or $\mathbb{F}_q^d$, such that

$$\sum_{\tau \in T} 1_\Omega(x - \tau) = 1 \ a.e.?$$

Fuglede Conjecture (1974-2003)

- In 1974, Fuglede conjectured that $L^2(\Omega)$, $\Omega \in \mathbb{R}^d$, possesses an orthogonal basis of exponentials iff Ω tiles $\mathbb{R}^d$ by translation.

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- The Fuglede conjecture is true for unions of three or fewer intervals in $\mathbb{R}$ (Laba and others). It is also true for convex sets in $\mathbb{R}^2$ (A.I., Katz and Tao, 2003) and convex sets in $\mathbb{R}^3$ (Greenberg and Lev, 2017).

Gabor bases

- A related question, introduced by Denes Gabor, is the following. For which $g \in L^2(\mathbb{R}^d)$ does there exist $S \subset \mathbb{R}^{2d}$ such that

$$\left\{ g(x-a)e^{2\pi i x \cdot b} \right\}_{(a,b) \in S} \text{ is an orthogonal basis ?}$$

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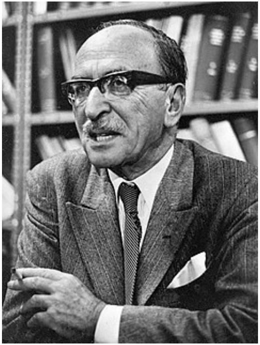

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Theorem

(A.I. and Mayeli (2018)) Let B_d denote the unit ball and $d \neq 1 \mod 4$. Then there **does not** *exist $S \subset \mathbb{R}^{2d}$ such that $\{1_{B_d}(x-a)\}_{(a,b)\in S}$ is an orthogonal basis of $L^2(\mathbb{R}^d)$.*

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- For general spectra, little is known even in the case when g is the indicator function of a symmetric convex set. In the non-symmetric case, orthogonal Gabor basis does not exist (Chung and Lai (2017)).

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- If E has positive Lebesgue measure, then $E - E$ contains an open ball, so $|\Delta(E)| > 0$. But what if E is much smaller?

The lattice construction

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- Observe that

$$|\Delta(E_{q_i}^s)| \lesssim q_i^{-\frac{d}{s}} \cdot \#\Delta(\mathbb{Z}^d \cap [0,q_i]^d).$$

The lattice construction (continued)

- In order to estimate $\#\Delta(\mathbb{Z}^d \cap [0, q_i]^d)$, observe that it is equivalent to count the number of values of

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- Going back, we see that

$$|\Delta(E_{q_i}^s)| \lesssim q_i^{-\frac{d}{s}} \cdot q_i^2 \to 0 \text{ if } s < \frac{d}{2}.$$

The Falconer Conjecture

Conjecture

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World records

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- Guth, A.I., Ou and Wang improved the threshold $\boxed{\frac{4}{3}}$ in $\mathbb{R}^2$ to $\boxed{\frac{5}{4}}$ (2018) and extended the result to other smooth metrics.

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- A natural question to ask is whether there is a direct quantitative connection between the Erdős and Falconer exponents. This turns out to be quite relevant in terms of applications to the theory of exponential bases.

Connecting Erdős and Fuglede

Theorem

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- i) A is *separated*, i.e $|a-a'| \geq c > 0$ for all $a \neq a' \in A$, and
- ii) A is *well-distributed*, i.e $\exists C > 0$ such that every cube of side-length C contains at least one point of A.

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- Since the zeroes of $J_{\frac{d}{2}}$ are uniformly separated, we conclude that

  $$\#\Delta(A \cap Q_R) \leq CR,$$

  which leads to an immediate contradiction in view of the Erdős exponents from the previous slide.

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- Here

$$K = \{x \in \mathbb{R}^d : \rho(x) \leq 1\}$$

and

$$\rho^*(\xi) = \sup_{x \in K} x \cdot \xi.$$

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- Can we obtain a contradiction by proving that the number of R^{-1}-separated elements of $\Delta_K(A_1, A_2)$ is much greater than CR?

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- ii) We must estimate the number of R^{-1}-separated distances.
- It turns out that once the Euclidean norm is replaced by the norm induced by a general convex body K and separated distances are required, the harmonic analysis techniques connected with the Falconer distance problem provides an efficient framework.

From Falconer to Erdős

- Suppose that one can show that if μ is a compactly supported Borel measure on $\mathbb{R}^d$, $d \geq 2$, with

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- Let P_n be a finite $n^{-\frac{1}{2}}$-separated point set in $[0,1]^d$, $d \geq 2$, of size n. Let

$$\mu_n(x) = n^{-1} n^{\frac{d}{s}} \sum_{p \in P} \phi(n^{\frac{1}{s}}(x-p)).$$

From Falconer to Erdős

- If we are fortunate and $I_s(\mu_n) \approx 1$, then

$$c < |\Delta_K(support(\mu_n))| \lesssim n^{-\frac{1}{s}} \mathcal{E}(P_n),$$

where $\mathcal{E}(P_n)$ is the number of $n^{-\frac{1}{s}}$-separated elements of $\Delta_K(E)$.

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- Various versions of this conversion mechanism were established by A.I.-Hofmann (2005), A.I.-Laba (2005) and A.I.-Rudnev-Uriarte-Tuero (2008).

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- By above, the result follows from the following Falconer variant:

Theorem

Let E, F be compact subsets of $\mathbb{R}^d$, $d \geq 2$, equipped with Borel measures μ_E, μ_F. Let K be a bounded symmetric convex set with a smooth boundary and non-vanishing curvature. Then

$$|\Delta_K(E,F)| \gtrsim \frac{1}{\sqrt{I_{\frac{d+1}{2}}(\mu_E) I_{\frac{d+1}{2}}(\mu_F)}}.$$

Death of Fuglede

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- Kolountzakis and Matolcsi (2006) obtained counter-examples in both directions in dimension 4 and 5 and Farkas (2006) obtained a counter-example in one direction in 3 dimensions.

Partial rebirth of Fuglede

Theorem

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- Suppose that $\{\chi(x \cdot a)\}_{a \in A}$ is an orthogonal basis for $L^2(E)$. Then $\#E = \#A$ (linear algebra).

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Lemma

(Magic Lemma) Suppose that $E \subset \mathbb{Z}_p^2$ such that

$$\widehat{1}_E(m) \equiv \sum \chi(-x \cdot m) 1_E(x) = 0 \text{ for some } m \in \mathbb{Z}_p^2.$$

Then $\widehat{1}_E(rm) = 0$ for all $r \neq 0$ and E is equidistributed on lines $\perp m$.

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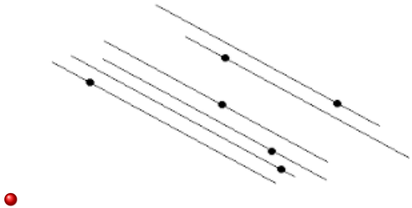
- It follows that $\#E = kp$. If $k = 1$ we see immediately that E must tile by translation since E has exactly one point on each line $\perp m$, for some $m \in \mathbb{Z}_p^2$. But how do we eliminate the case $k > 1$?

Basis implies tiling (continued)



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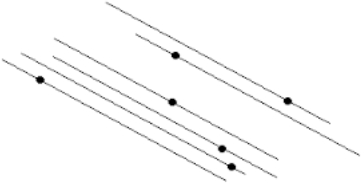

- If $k > 1$, $\#E = \#A > p$, so A determines every possible direction, i.e every non-zero $x \in \mathbb{Z}_p^2$ can be written in the form

$$t(a - a'), \; a, a' \in A, \; t \in \mathbb{Z}_p.$$

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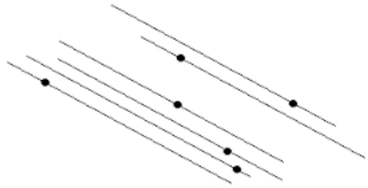

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$$t(a - a'), \;\; a, a' \in A, \;\; t \in \mathbb{Z}_p.$$

- By orthogonality and the magic lemma, for any $a \neq a' \in A, r \neq 0$,

$$\sum_x \chi(x \cdot r(a - a'))1_E(x) = 0$$

and we conclude that $E = \mathbb{Z}_p^2$. This proves that E must tile.

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- Suppose that E tiles by translation, i.e for all $x \in \mathbb{Z}_p^2$,

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- This implies that either $T = \mathbb{Z}_p^2$ (not interesting), or there exists m such that $\widehat{1}_E(m) = 0$ and the magic lemma applies and we deduce that E is equidistributed on the p lines $\perp m$.

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- After applying a rotation we may assume that $E = \{(t, 0) : t \in \mathbb{Z}_p\}$ and it is easy to see that taking $A = E$ gives us an orthogonal exponential basis.
- This completes the proof up to the verification of the Magic Lemma.

Proof of the magic lemma

- Suppose that $\widehat{1}_E(m) = 0$ for some $m \neq (0,0)$. Then

$$0 = \sum_{x \in \mathbb{Z}_p^d} \chi(-x \cdot m) 1_E(x) = \sum_{t \in \mathbb{Z}_p} (\chi(-1))^t n(t),$$

where

$$n(t) = \sum_{x \cdot m = t} 1_E(x).$$

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- This implies that $n(t)$ is constant in t by the uniqueness of the minimal polynomial, so E is equidistributed on lines $\perp m$.

- *THANK YOU*!

Shannon, Simon and I

Simon and I

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Josh, Nathan and I