Harmonic measure with lower dimensional boundaries

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Harmonic measure in the "classical case"

Here and below $\Omega \subset \mathbb{R}^n$ is a connected open set and $E = \partial\Omega$.

For the next definitions, assume Ω bounded. Let $X \in \Omega$ be given.

Intuitive definition of the harmonic measure $\omega^X = \omega^X_\Omega$: for $A \subset \partial\Omega$ measurable, $\omega^X_\Omega(A)$ is the probability of a Brownian path starting from X, to lie in A the first time it hits $E = \partial\Omega$.

This works on many domains (if E is not too small).

Analytic definition : Find a way to solve, for $g \in \mathcal{C}(\partial\Omega)$, the Dirichlet problem $\Delta u = 0$ on Ω and $u = g$ on $\partial\Omega$.

Then use the maximum principle ($|u(X)| \leq ||g||_\infty$) and Riesz to find a probability measure ω^X_Ω such that

$$u(X) = \int_E g(\xi) d\omega^X_\Omega(\xi) \quad \text{for } g \in \mathcal{C}(\partial\Omega).$$

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Ahlfors regular sets

We'll restrict to Ahlfors regular (AR) sets of dimension $d < n$.
That is, $E \subset \mathbb{R}^n$ closed and such that for some Borel measure σ on E and some $C_0 \geq 1$,

$$(1) \qquad C_0^{-1} r^d \leq \sigma(E \cap B(x,r)) \leq C_0 r^d \quad \text{for } x \in E \text{ and } r > 0.$$

So now we take E unbounded.
Easy to show: $C^{-1}\mathcal{H}^d_{|E} \leq \sigma \leq C\mathcal{H}^d_{|E}$. So we may take $\sigma = \mathcal{H}^d_{|E}$.

Convention for below: classical case is when $d = n - 1$ and high co-dimension case when $d < n - 1$.

"Main" question since we have σ: when, in geometric terms for E, is ω_Ω^X absolutely continuous with respect to σ?
By Harnack, $\omega_\Omega^X << \omega_\Omega^Y << \omega_\Omega^X$ for $X \neq Y$, so the question does not depend much on X.
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Absolute Continuity (in the classical case)

A subject with a long history. Just hints here.

When $n = 2$, $\mathbb{R}^2 \simeq \mathbb{C}$, ω_Ω has conformal invariance, the question is related to a control of $|\psi'|$ for conformal mappings ψ; important results by Riesz & Riesz (1916), Lavrentiev (1936), Carleson, Makarov, Jones, Bishop, etc.

For $n \geq 2$, Dahlberg (77), Jerison-Kenig, Wolff, Bourgain, D.-Jerison, Semmes (up to A_∞ for $NTA + UR$)...

More recently Hofmann, Martell, Uriarte-Tuero ; Azzam, Nyström, Toro (converse). Add Bennewitz, Lewis, Mourgoglou, Tolsa, etc. for optimal replacements for NTA.
See A-H-M-M-Mayboroda-T-Volberg, for non AR boundaries. Etc.
Just one result here: if Ω and its complement satisfy the Corkscrew conditions and Ω satisfies the Harnack chain condition (some reasonable connectedness conditions) and $E = \partial\Omega$ is AR of dimension $n - 1$, then

$$\omega \in A_\infty(\sigma) \Leftrightarrow E \text{ is uniformly rectifiable (Explain a bit).}$$

Definitions of Corkscrew points and of $A_\infty(\sigma)$

The Corkscrew property says that for $x \in E$ and $r > 0$ we can choose a CS point $A_{x,r} \in \Omega \cap B(x, r)$, with $\mathrm{dist}(A_{x,r}, \partial\Omega) \geq C_{cs}^{-1} r$. We do so.
The Harnack chain condition is the existence of thick paths in Ω from any $X \in \Omega$ to any $Y \in \Omega$. Not written here today.

Our preferred condition of quantitative mutual absolute continuity is $\omega \in A_\infty(\sigma)$, which means:
for $\varepsilon > 0$ there exists $\delta > 0$ (small) such that, for $A \subset E \cap B(x, r)$ (a measurable set and a ball centered on E),

$$\omega_X^{A_{x,r}}(E) < \delta \Longrightarrow \frac{|E|}{|B(x,r)|} < \epsilon, \tag{2}$$

Really looks like standard A_∞. A symmetric relation here (because ω and σ are doubling). Other equivalent definitions exist.

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Classical elliptic operators

The results above extend to the situation where Δ is replaced by an ellipitic operator $L = -\operatorname{div}[A(X)\nabla]$, where the $n \times n$ matrix-valued function A with real coefficients satisfies the usual boundedness and ellipticity conditions

$$(2) \qquad \big|A(X)\xi \cdot \zeta\big| \leq C_e|\xi||\zeta| \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n,$$

$$(3) \qquad A(X)\xi \cdot \xi \geq C_e^{-1}|\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

The main results above (including some converse!) extend to such $L = -\operatorname{div}[A(X)\nabla]$, under appropriate Carleson measure conditions on the size/variations of A.

Works by Azzam, Hofmann, Martell, Mayboroda, Nyström, Pipher, Toro just for this; I forget many and concerning close problems. Important here because we'll not have a beautiful Laplacian soon.

Higher codimensions?

Why take $E = \partial\Omega$ of dimension $d < n-1$?

Curiosity: Do some things go through? What is so special about codimension 1? What is the relation of the Riesz transform?

Laziness: we don't need to understand (weaker versions of) the Corkscrew and Harnack conditions, because these are "trivially" true when $E \in AR(d)$.

After the fact: discover new operators.

For me at least: the pleasure to do the forbidden thing (see below).

From now on $E = \partial\Omega$, but now $E \in AR(d)$ for some $d < n-1$. $\sigma = \mathcal{H}^d_{|E}$ (for instance). By the way, $d \in \mathbb{N}$ is not needed.

As soon as $d \leq n-2$, small problem with Δ and the other elliptic operators L: our definitions of harmonic measure fail because

- Since E is too small (polar), the Brownian paths do not meet E.

- We cannot solve the Dirichlet problem for continuous functions on $\Omega = \mathbb{R}^n \setminus E$; nice harmonic functions on Ω extend through E!

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Our program

Two (recent independent) ways to deal with this.

- J. Lewis, K. Nyström, A. Vogel : Replace Δ with a p-Laplacian (a non linear operator);

- DFM: Replace Δ with a degenerate (but linear) elliptic operator L, with coefficients that tend to ∞ near E.

This is what we'll describe here.
Linear operators, but which depend on E!
Need to reconstruct elliptic theory before we think about $A_\infty(\sigma)$.
The simplest case is when $E = \mathbb{R}^d \subset \mathbb{R}^n$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$; then

$$L = -\operatorname{div}[A(X)\nabla] \text{ with } A(X) = \operatorname{dist}(X, E)^{d+1-n}\, \mathbf{I_n}$$

is very good; radial solutions of $Lu = 0$ in Ω come from harmonic solutions of $\Delta u = 0$ in $\mathbb{R}^{d+1}_+$.
Think of the weight $w(X) = \operatorname{dist}(X, E)^{d+1-n}$ as inducing a drift that on average pushes Brownian motion towards E?

Our (degenerate) elliptic operators (1): The geometry

Looking for the right class... We'll use the approach of minimizing energy in the right homogeneous Sobolev space W. Set

$$\delta(X) = \mathrm{dist}(X, E) \ \text{ for } X \in \Omega.$$

Take the weight $w(X) = \mathrm{dist}(X, E)^{d+1-n} = \delta(X)^{d+1-n}$ on Ω. (In the classical case, this would yield $w = 1$.) Define

$$(4) \quad W = \mathcal{H}^1(\Omega) = \Big\{ u \in W^{1,2}_{loc}(\Omega)\, ; \int_\Omega |\nabla u|^2 w(X) dX < +\infty \Big\}.$$

And on $E = \partial\Omega$, define the space $H = H^{1/2}(E)$ by

(5)

$$H = \Big\{ f \in L^2_{loc}(E)\, ; \int_E \int_E \frac{|f(x) - f(y)|^2 d\sigma(x) d\sigma(y)}{|x-y|^{d+1}} < +\infty \Big\}.$$

Since $E \in AR(d)$, we can construct a trace operator $Tr = W \to H$ and an extension operator $Ex : H \to W$, with $Tr \circ Ex = I$.

Degenerate elliptic operators (2): Our class of matrices, Forms, and Lax-Milgram

The natural class of operators that goes with W is the class of operators $L = -\operatorname{div}[A(X)\nabla]$ such that $A(X) = w(X)\mathcal{A}(X)$ for some classical elliptic matrix $\mathcal{A}$ [i.e., that satisfies (2) and (3)]. We'll write this $A \in \mathcal{M}_{dell}(\Omega)$ or even $L \in \mathcal{M}_{dell}(\Omega)$.

Because then, the form $(u, v) \to \int_\Omega A\nabla u \cdot \nabla v$ is naturally defined for $u, v \in W$. And by the Lax-Milgram theorem, the problem

(6) $$Lu = 0 \text{ (weakly on } \Omega) \text{ and } Tr(u) = g$$

has a unique solution $u \in W$ for any $g \in H$.

Then there is a long list of things to check (see below), following some classical proofs, to define harmonic measures ω_L^X, $X \in \Omega$, and an acceptable elliptic theory for $L \in \mathcal{M}_{dell}(\Omega)$. But...

Elliptic Theory (1): A statement

Recall we have a class $\mathcal{M}_{dell}(\Omega)$ of matrices A, or operators $L = -\operatorname{div}[A(X)\nabla]$. By acceptable elliptic theory we mean...

Theorem (DMS)

Let $E \subset \mathbb{R}^n$ be an Ahlfors regular set of dimension $d < n-1$ (d is an integer or not).
Take $L = \mathrm{divA}\nabla$ such that $\mathrm{dist}(X,E)^{n-d-1}A(X)$ is bounded elliptic. That is, let $L \in \mathcal{M}_{dell}(\Omega)$.
Then for $f \in C_0(E)$ we can solve $Lu = 0$ on Ω, with $Trace(u) = f$ on E, and for $X \in \Omega$, and there is a doubling probability measure $\omega^X = \omega_L^X$ on E such that $u(X) = \int_E f d\omega^X$ for f and u as above.
And with additional properties listed below.

We call ω^X or ω_L^X the harmonic measure. When X changes, these measures are mutually absolutely continuous (by Harnack).
See below for additional properties. In short, ω^X has the "usual properties" valid for NTA domains in the co-dimension 1 case.

Elliptic Theory: the main differences

Good news: the existence of corkscrew points for Ω, and the Harnack chain condition, follow from the low dimension ($E \in AR(d)$ for some $d < n-1$).

Looks dangerous: no barrier functions , in fact no complementary component. We replace this with the weighted Poincaré estimate

$$(7) \qquad \int_{B(x,r)} |u(X)|^2 w(X) dX \leq C r^2 \int_{B(x,r)} |\nabla u(X)|^2 w(X) dX$$

when $x \in E$, $r > 0$ and $u \in W$ (or smoother) is such that $Tr(u) = 0$ on $E \cap B$.
Which is proved essentially as the boundedness of Tr (well chosen w and we can find lots of paths from $X \in \Omega \cap B$ to $y \in E \cap B$).

Elliptic Theory (the shopping list)

We...
- Check by hand the existence of Harnack chains (even tubes);
- Prove some weighted Poincaré inequalities in Ω;
- Define weak solutions;
- Prove interior Moser estimates, interior Hölder estimates, Harnack inequalities for solutions;
- Prove Caccioppoli, Moser, Harnack, Hölder estimates on the boundary;
- Use this to prove a maximum principle, solve the Dirichlet problem, and define the harmonic measure ω_L^X;
- Define Green functions and estimate them;
- Prove comparison principles (at the boundary) and ω_L is doubling.

At the end we get the usual toolbox, and are ready for more refined questions where additional regularity for E and A should be needed.

Specific operators

What should replace Δ (or good perturbations of Δ) here?
We'll prefer operators $L = -\operatorname{div}[A(X)\nabla]$ where $A(X) = g(X)\,\mathbf{I_n}$ for some positive function g.
Given the definition of $\mathcal{M}_{dell}$, the simplest is to take $g(X) = \mathrm{dist}(X, E)^{d+1-n} = \delta(X)^{d+1-n}$, but for $d \geq 2$ we think $\delta(X)$ is not smooth enough, so we prefer

$$(8) \qquad L = -\operatorname{div}[A(X)\nabla] \quad \text{with} \quad A(X) = D_\alpha(X)^{-n+d+1}.$$

where we pick $\alpha > 0$ and set

$$(9) \quad D_\alpha(X) = R_\alpha(X)^{-1/\alpha} \quad \text{with} \quad R_\alpha(X) = \int_E |X-y|^{-d-\alpha} d\sigma(y).$$

Think that D_α is a smoother substitute for δ, defined as intrinsically as we could. Homogeneity is correct.
$A \in \mathcal{M}_{dell}(\Omega)$ and we would get $L = c\Delta$ when $E = \mathbb{R}^d \subset \mathbb{R}^{d+1}$.
A is as nice as we can, and we expect L to be good when E is nice.

A quantitative mutual absolute continuity statement

Question: Given Ω and $E = \partial\Omega$, set D_α and $L = L_\alpha$ as above. When does $\omega_L \in A_\infty(\sigma)$?

In co-dimension 1, this would be when E is uniformly rectifiable.

Here is an analogue of Dahlberg's theorem (77). Take $E = \left\{x + \varphi(x)\,;\, x \in \mathbb{R}^d\right\}$, where $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is ε-Lipschitz. Set $\sigma = \mathcal{H}^d_{|E}$, $D = D_\alpha$ as in (9), $L = -\operatorname{div}[D_\alpha^{-n+d+1}\nabla]$ as in (8).

THEOREM (DMS)

If ε is small enough, then $\omega_L \in A_\infty(\sigma)$.

Works also with $A(X) = \delta(X)^{-n+d+1}$ when $d = 1$ (but for $d > 1$ $\delta(X)$ is too irregular for our proof.

Comments and extensions

1. The Dirichlet problem for L^p, p large, for these operators was solved by J. Feneuil, S. Mayboroda, and Z. Zihui [Extra complications for localizations].

2. Some different operators work too ("Carleson" perturbations of L). With little extra work in this case.

Proof of the thm: treat enough degenerate elliptic operators in the special case of $E = \mathbb{R}^d \subset \mathbb{R}^n$, and then change variables correctly. Need to be careful about treating the last $n - d$ coordinates in the change of variable almost isotropically; see more below.

Small Lipschitz required for our change of variables to be bilipschitz. Otherwise, we would need a different formula, or to cut the change of variables into smaller pieces and do as above?

Naturally expected: E uniformly rectifiable is enough. See below.

Good Degenerate Elliptic Operators in $\Omega_0 = \mathbb{R}^n \setminus \mathbb{R}^d$

Here we take $E = \mathbb{R}^d \subset \mathbb{R}^n$ $(d \leq n-2)$, write the variable $(x,t) \in \mathbb{R}^n \times \mathbb{R}^{n-d}$, and take $A(x,t) = |t|^{-n+d+1} M(x,t)$, with

$$M(x,t) = \begin{pmatrix} M^1(x,t) & \mathcal{C}^2(x,t) \\ \mathcal{C}^3(x,t) & b(x,t) I_{n-d} + \mathcal{C}^4(x,t) \end{pmatrix}, \tag{10}$$

where $M^1(X)$ is an elliptic $d \times d$ matrix, the $|\mathcal{C}^j|^2$ satisfy Carleson measure estimates, $\mathbf{I}_{n-d}$ is the identity matrix, and b is a function on $\Omega_0 = \mathbb{R}^n \setminus \mathbb{R}^d$ such that $C^{-1} \leq b \leq C$ and $|t|^2 |\nabla b|^2$ satisfies a Carleson measure estimate.

Theorem

Under these assumptions, ω^X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, with an A_∞ density.

Comments: 1. General enough even when $d = n-1$.

2. Notice the isotropy in t.

3. More good operators here = more OK change of variables later.

Proofs

Main steps in the proof of the theorem in $\mathbb{R}^d$:
- An estimate like $||Su||_2^2 \leq C||Nu||_2^2$ for solutions (localized), obtained by integrations by parts and Carleson measure estimates;
- As a consequence, Carleson measure estimates on the square function of solutions of the Dirichlet problem with bounded boundary values on $\mathbb{R}^d$;
- A general argument, roughly as in Kenig-Kirchheim-Pipher-Toro or Dindos-Petermichl-Pipher, to go from there to A_∞.

And then, for the theorem in $\Omega = \mathbb{R}^n \setminus E$, an appropriate bi-Lipschitz change of variable $\psi : \Omega_0 \to \Omega$, and some regularity for D (coming from Tolsa α-numbers for σ).

It is important that ψ almost preserves the directions orthogonal to $\mathbb{R}^d$ or E. Just moving points up and down, even smoothly, is not good enough to give the right form to the conjugate operator of L.

This explain the need for a specially designed change of variables.

Uniform Rectifiability Dream Part 1. (D.-Mayboroda)

Modulo writing up (more?), we sort of claim the following.

Theorem

Let $E \subset \mathbb{R}^n$ be Ahlfors regular of dimension $d < n-1$ and uniformly rectifiable.
Take $L = \mathrm{div} A \nabla$, with $A(X) = D_\alpha(X)^{-n+d+1} I_n$ as in (8)–(9).
Then $\omega_L \in A_\infty(\mathcal{H}^d_{|E})$.

What we hoped for before we started!
Notice again, no NTA condition is added; it is automatic.

Proof: The needed ingredients are (at least) a stopping time as in the corona construction, a good change of variable for each (corona) region as in the Reifenberg parameterization of D.-Toro, an extrapolation result that shows that a control in every region is enough, a generalization by DJF of the elliptic theory above to accommodate boundaries of mixed dimensions, and a little bit more time please.

Questions, looking for a converse

Again this should stay true with $D(X) = \mathrm{dist}(X, E)$ when $d = 1$. In general we believe but can't prove that the additional regularity of D is needed. Carleson measure estimates on the variations of D are useful in the proof.

Notice the nonlinear nonlocal dependence on E through D_α.

But is there a converse? That is, take D_α and assume that $\omega_L \in A_\infty(\sigma)$; can we say that E is uniformly rectifiable?
Can $\omega_L \in A_\infty(\sigma)$ ever happen when $d \notin \mathbb{N}$?

OK but quite hard in codimension 1, with the help of the boundedness of the Riesz transform.

A statement of order 0, corresponding to a blow-up limit at a point of density of $\frac{d\omega}{d\sigma}$ would be nice (suppose $\omega^\infty = \sigma$; is E flat?).

So far, we have no Riesz transform that would play the same role (at least for describing the solutions). We can define some square functions (as in the USFE), but see no obvious relation with ω_L^X.

A strange case (D.-Engelstein-Mayboroda)

We found a special case where things are simpler, and expected that the converse would be easier to prove in this case, i.e., when

$$\alpha = n - d - 2 \quad \text{(hence } n > d + 2\text{, but } d \notin \mathbb{N} \text{ allowed.}$$

Then $R_\alpha(X) = \int_E \frac{d\sigma(y)}{|X - y|^{n-2}}$ is harmonic, and a computation shows that $D(x) = R(x)^{-1/\alpha}$ satisfies $Lu = 0$.

Assume to simplify that E is rectifiable. By another computation, $\frac{\partial D}{\partial n} = 1$ (the normal derivative, maybe multiply σ by a constant to normalize) a.e. on E, and so D is the Green function for L, with pole at ∞. That is, $\omega^\infty = \sigma$.

Very strange, no? (Think about the Brownian paths.)

Even when E is merely Ahlfors regular, it seems that $\omega << \sigma$, with a density h such that $C^{-1} \leq h \leq C$.

Not at all what we expected! Does $\alpha = n - d - 2$ mean something special? Is it the only case where this happens?

[All this is subject to writing down, but it is coming.]

More operators to study

Along the process we found out that there are other natural questions in this context. Such as:

Improve the theory for the operators we have (We've seen the beginning of Dirichlet problem on L^p, many others...)

Extend again our class $\mathcal{M}_{dell}$, in particular to the case when the geometry of E comes with a doubling measure μ (nice enough if we want $A_\infty(\mu)$)?

For instance that would allow us to deal with sawtooth domains (two dimensions at the same time).

Or in connection to fractional integration?

Did I mention connections to Square Function Estimates and the search for a relevant Riesz transforms?

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Conclusion

Thanks for your attention!

HAPPY BIRTHDAY STEVE!