The Neumann problem for symmetric higher order elliptic differential equations

Ariel Barton

Joint work with Steve Hofmann and Svitlana Mayboroda

May 30, 2018

Workshop on Real Harmonic Analysis and its Applications to Partial Differential Equations and Geometric Measure Theory: on the occasion of the 60th birthday of Steve Hofmann

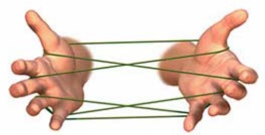
ICMAT, Madrid (Spain)

Second order differential equations: $\Delta = \partial_{xx} + \partial_{yy} + \dots$

The force required to bend a string under tension is proportional to the second derivative of its displacement, $\partial_{xx} h$.

$\partial_{tt} h = c\partial_{xx} h$

$\partial_{xx} h = c\rho$

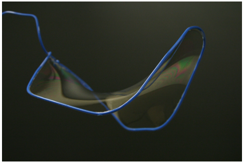


$\partial_{xx} h = 0$

The force required to bend a membrane under tension is proportional to $\Delta h = \partial_{xx} h + \partial_{yy} h$.

$\partial_{tt} h = c\Delta h$

$\Delta h = c\rho$

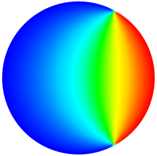


$\Delta h = 0$

Harmonic boundary value problems

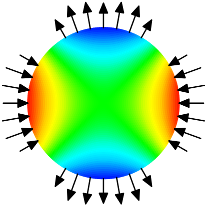
There is an extensive theory for the harmonic Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$



and the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \nu \cdot \nabla u = g & \text{on } \partial\Omega. \end{cases}$$

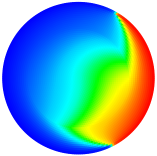


Second order boundary value problems

Suppose the matrix A is uniformly positive definite and bounded:

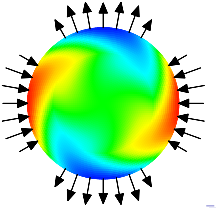
$$\operatorname{Re} \overline{\vec{v}} \cdot A(X)\vec{v} \geq \lambda|\vec{v}|^2, \qquad |A(X)| \leq \Lambda \quad \text{for all } X \in \mathbb{R}^d,\ \vec{v} \in \mathbb{C}^d.$$

There is an extensive theory for the second order elliptic Dirichlet problem



$$\begin{cases} \nabla \cdot A\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

and the Neumann problem

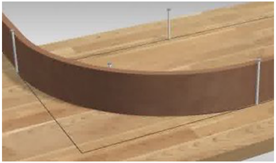


$$\begin{cases} \nabla \cdot A\nabla u = 0 & \text{in } \Omega, \\ \nu \cdot A\nabla u = g & \text{on } \partial\Omega. \end{cases}$$

Higher order differential equations

The force required to bend a thin elastic rod is proportional to the fourth derivative of its displacement, $\partial_{xxxx} h$.

The force required to bend a thin elastic plate is proportional to $\Delta^2 h = \partial_{xx}(\partial_{xx} h) + \partial_{xy}(2\partial_{xy} h) + \partial_{yy}(\partial_{yy} h)$.



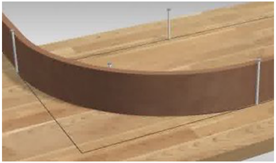
Higher order differential equations

The force required to bend a thin elastic rod is proportional to the fourth derivative of its displacement, $\partial_{xxxx} h$.



The force required to bend a thin elastic plate is proportional to
$\Delta^2 h = \partial_{xx}(\partial_{xx} h) + \partial_{xy}(2\partial_{xy} h) + \partial_{yy}(\partial_{yy} h)$.





(Euler-Bernoulli beam equation) The force required to bend an inhomogeneous thin elastic rod is proportional to the fourth derivative of its displacement $\partial_{xx}(E(x)\, I(x)\, \partial_{xx} h)$.

Higher order boundary value problems

We are interested in higher-order differential equations such as the biharmonic equation (in $\mathbb{R}^d$)

$$\Delta^2 u = \nabla^2 \cdot \nabla^2 u = \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk}(\partial_{jk} u) = 0$$

or more generally

$$\nabla^m \cdot A \nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0.$$

Higher order boundary value problems

We are interested in higher-order differential equations such as the biharmonic equation (in $\mathbb{R}^d$)

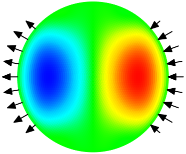
$$\Delta^2 u = \nabla^2 \cdot \nabla^2 u = \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk}(\partial_{jk} u) = 0$$

or more generally

$$\nabla^m \cdot A \nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0.$$

We are interested in the Dirichlet problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f, \ \nu \cdot \nabla u = g & \text{on } \partial\Omega. \end{cases}$$



Higher order boundary value problems

We are interested in higher-order differential equations such as the biharmonic equation (in $\mathbb{R}^d$)

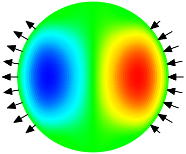
$$\Delta^2 u = \nabla^2 \cdot \nabla^2 u = \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk}(\partial_{jk} u) = 0$$

or more generally

$$\nabla^m \cdot A \nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0.$$

We are interested in the Dirichlet problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \nabla u = \vec{f} & \text{on } \partial\Omega. \end{cases}$$



Higher order boundary value problems

We are interested in higher-order differential equations such as the biharmonic equation (in $\mathbb{R}^d$)

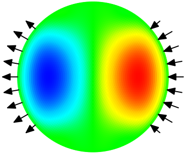
$$\Delta^2 u = \nabla^2 \cdot \nabla^2 u = \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk}(\partial_{jk} u) = 0$$

or more generally

$$\nabla^m \cdot A \nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0.$$

We are interested in the Dirichlet problem

$$\begin{cases} \nabla^m \cdot A \nabla^m u = 0 & \text{in } \Omega, \\ \nabla^{m-1} u = \dot{f} & \text{on } \partial\Omega. \end{cases}$$



Higher order Neumann boundary values

In the second-order case $\nabla \cdot A\nabla u = 0$, the Neumann boundary values of u are $\nu \cdot A\nabla u$.

Higher order Neumann boundary values

In the second-order case $\nabla \cdot A\nabla u = 0$, the Neumann boundary values of u are $\nu \cdot A\nabla u$.
Notice that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u$$

depends only on $\nabla^{m-1}\varphi\big|_{\partial\Omega}$.

Higher order Neumann boundary values

In the second-order case $\nabla \cdot A\nabla u = 0$, the Neumann boundary values of u are $\nu \cdot A\nabla u$.
Notice that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u$$

depends only on $\nabla^{m-1}\varphi\big|_{\partial\Omega}$. So

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u = \int_{\partial\Omega} \nabla^{m-1}\varphi \cdot \dot{M}_\Omega^A u \, d\sigma$$

for some $\dot{M}_\Omega^A u$.

Higher order Neumann boundary values

In the second-order case $\nabla \cdot A\nabla u = 0$, the Neumann boundary values of u are $\nu \cdot A\nabla u$.

Notice that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u$$

depends only on $\nabla^{m-1}\varphi\big|_{\partial\Omega}$. So

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u = \int_{\partial\Omega} \nabla^{m-1}\varphi \cdot \dot M_\Omega^A u \, d\sigma$$

for some $\dot M_\Omega^A u$.

If $m = 1$ then $M_\Omega^A u = \nu \cdot A\nabla u$.

Higher order Neumann boundary values

In the second-order case $\nabla \cdot A\nabla u = 0$, the Neumann boundary values of u are $\nu \cdot A\nabla u$.

Notice that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u$$

depends only on $\nabla^{m-1}\varphi\big|_{\partial\Omega}$. So

$$\int_\Omega \nabla^m \varphi \cdot A\nabla^m u = \int_{\partial\Omega} \nabla^{m-1}\varphi \cdot \dot{M}_\Omega^A u \, d\sigma$$

for some $\dot{M}_\Omega^A u$.

If $m = 1$ then $M_\Omega^A u = \nu \cdot A\nabla u$.

A free boundary corresponds to $\dot{M}_\Omega^A u = 0$.

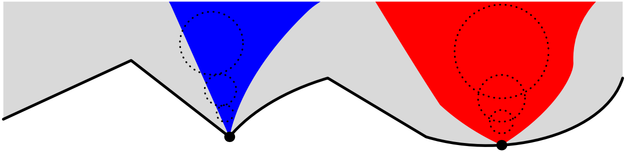
Higher order boundary value problems

We are interested in the Dirichlet problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \\ \|\widetilde{N}(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}, \end{cases}$$

and the Neumann problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \dot{M}_\Omega^A u = \dot{g}, \\ \|\widetilde{N}(\nabla^m u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{L^p(\partial\Omega)}. \end{cases}$$



$$Nu(X) = \sup\{|u(Y)| : |X-Y| < (1+a)\,\mathrm{dist}(Y,\partial\Omega)\}$$

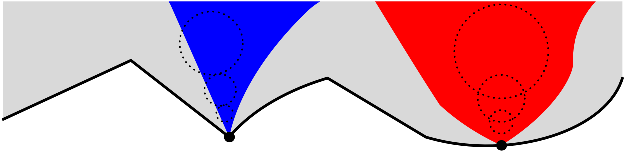
Higher order boundary value problems

We are interested in the Dirichlet problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \\ \|\widetilde{N}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}, \end{cases} \qquad \begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \\ \|\widetilde{N}(\nabla^{m}u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}, \end{cases}$$

and the Neumann problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \dot{M}^A_\Omega u = \dot{g}, \\ \|\widetilde{N}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{\dot{W}^p_{-1}(\partial\Omega)}, \end{cases} \qquad \begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \dot{M}^A_\Omega u = \dot{g}, \\ \|\widetilde{N}(\nabla^{m}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{L^p(\partial\Omega)}. \end{cases}$$



$$Nu(X) = \sup\{|u(Y)| : |X-Y| < (1+a)\operatorname{dist}(Y,\partial\Omega)\}$$

Regularity of coefficients

(Caffarelli, Fabes, Kenig, 1981) There is a real, symmetric matrix A, continuous in $B \subset \mathbb{R}^2$, such that

$$\nabla \cdot A\nabla u = 0 \text{ in } B, \qquad u = f \text{ on } \partial B, \qquad \|Nu\|_{L^p(\partial B)} \lesssim \|f\|_{L^p(\partial B)}$$

is ill-posed for all $1 < p < \infty$.

Regularity of coefficients

(Caffarelli, Fabes, Kenig, 1981) There is a real, symmetric matrix $\widetilde{A}$, continuous in $B \subset \mathbb{R}^2$, such that

$$\nabla \cdot \widetilde{A} \nabla u = 0 \text{ in } B, \quad \nu \cdot \widetilde{A} \nabla u = g \text{ on } \partial B, \quad \|N(\nabla u)\|_{L^p(\partial B)} \lesssim \|g\|_{L^p(\partial B)}$$

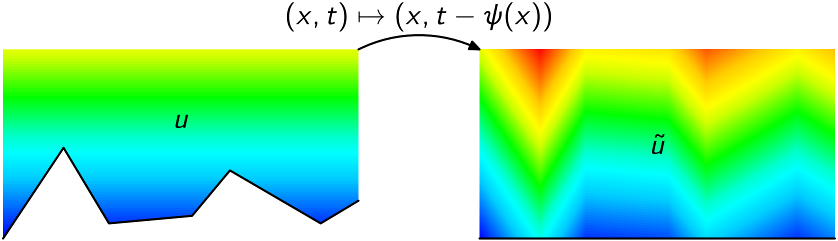
is ill-posed for all $1 < p < \infty$.

Regularity of coefficients

(Caffarelli, Fabes, Kenig, 1981) There is a real, symmetric matrix $\widetilde{A}$, continuous in $B \subset \mathbb{R}^2$, such that

$$\nabla \cdot \widetilde{A} \nabla u = 0 \text{ in } B, \quad \nu \cdot \widetilde{A} \nabla u = g \text{ on } \partial B, \quad \|N(\nabla u)\|_{L^p(\partial B)} \lesssim \|g\|_{L^p(\partial B)}$$

is ill-posed for all $1 < p < \infty$.



If $\Delta u = 0$, then $\nabla \cdot A_\psi \nabla \tilde{u} = 0$, where

$$A_\psi(x,t) = \begin{pmatrix} I & \nabla\psi(x) \\ \nabla\psi(x)^T & 1 + |\nabla\psi(x)|^2 \end{pmatrix}$$

Notice $A_\psi(x,t)$ is real, symmetric, and t-independent.

t-independence and Lipschitz domains

From now on we will work with equations of the form

$$\nabla^m \cdot A\nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta}\partial^\beta u) = 0$$

where the coefficient matrix A is elliptic and t-independent, that is,

$$A(x,t) = A(x,s) = A(x) \quad \text{for all } x \in \mathbb{R}^{d-1} \text{ and all } s,\ t \in \mathbb{R}.$$

t-independence and Lipschitz domains

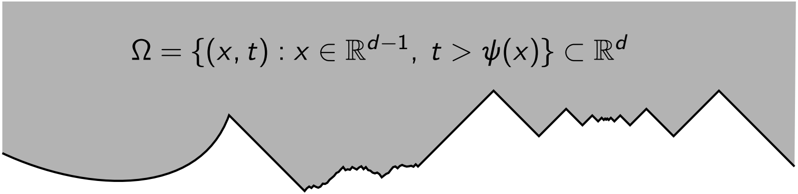
From now on we will work with equations of the form

$$\nabla^m \cdot A \nabla^m u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0$$

where the coefficient matrix A is elliptic and t-independent, that is,

$$A(x,t) = A(x,s) = A(x) \quad \text{for all } x \in \mathbb{R}^{d-1} \text{ and all } s,\ t \in \mathbb{R}.$$

We will work in Lipschitz graph domains



where $\nabla\psi \in L^\infty(\mathbb{R}^{d-1})$.

History: the second-order case

(Jerison and Kenig, 1981) If A is real-valued, t-independent and symmetric, then for all $2-\varepsilon < p < \infty$ we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

History: the second-order case

(Jerison and Kenig, 1981) If A is real-valued, t-independent and symmetric, then for all $2-\varepsilon < p < \infty$ we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

(Kenig and Pipher, 1993) If A is t-independent, real-valued and symmetric, and if $1 < p < 2+\varepsilon$, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau f\|_{L^p(\partial\Omega)},$$

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad \nu \cdot A\nabla u = g, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

History: the second-order case

(Jerison and Kenig, 1981) If A is real-valued, t-independent and symmetric, then for all $2-\varepsilon < p < \infty$ we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

(Kenig and Pipher, 1993) If A is t-independent, real-valued and symmetric, and if $1 < p < 2+\varepsilon$, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau f\|_{L^p(\partial\Omega)},$$

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad \nu \cdot A\nabla u = g, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

(Auscher and Mourgoglou, 2014) If A is t-independent, real-valued and symmetric, and if $2-\varepsilon < p < \infty$, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad \nu \cdot A\nabla u = g, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|g\|_{\dot W^{-1,p}(\partial\Omega)}.$$

History: the second-order case

(Kenig, Koch, Pipher, Toro, 2000) If $\Omega \subset \mathbb{R}^2$, $\frac{1}{\varepsilon} < p < \infty$, and A is real, t-independent, but not symmetric, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

(Kenig and Rule, 2009) If $\Omega \subset \mathbb{R}^2$, and if $1 < p < 1 + \varepsilon$, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau f\|_{L^p(\partial\Omega)},$$

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad \nu \cdot A\nabla u = g, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

History: the second-order case

(Kenig, Koch, Pipher, Toro, 2000) If $\Omega \subset \mathbb{R}^2$, $\frac{1}{\varepsilon} < p < \infty$, and A is real, t-independent, but not symmetric, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

(Kenig and Rule, 2009) If $\Omega \subset \mathbb{R}^2$, and if $1 < p < 1 + \varepsilon$, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau f\|_{L^p(\partial\Omega)},$$

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad \nu \cdot A\nabla u = g, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

(Hofmann, Kenig, Mayboroda, Pipher, 2015) If A is not symmetric, then we can solve

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)},$$

$$\nabla \cdot A\nabla u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau f\|_{L^p(\partial\Omega)}.$$

History: the higher-order case

(Dahlberg, Kenig, Verchota, 1986) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

History: the higher-order case

(Dahlberg, Kenig, Verchota, 1986) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

(Verchota, 1990) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

History: the higher-order case

(Dahlberg, Kenig, Verchota, 1986) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

(Verchota, 1990) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

(Pipher and Verchota, 1995) If Ω is a bounded Lipschitz domain and A is constant, and if $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problems

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \ \|N(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)},$$

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \ \|N(\nabla^m u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

History: the higher-order case

(Dahlberg, Kenig, Verchota, 1986) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

(Verchota, 1990) If Ω is a bounded Lipschitz domain and $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

(Pipher and Verchota, 1995) If Ω is a bounded Lipschitz domain and A is constant, and if $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problems

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \; \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \; \|N(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)},$$

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \; \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \; \|N(\nabla^{m} u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

(Verchota, 2005) If $2-\varepsilon < p < 2+\varepsilon$ then we can solve the biharmonic Neumann problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \dot{M}_A^\Omega u = \dot{g}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{L^p(\partial\Omega)}.$$

History: the higher-order case

(Verchota, 1996) If $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $2 - \varepsilon < p < \infty$, then we can solve the problem

$$\nabla^m \cdot A \nabla^m u \text{ in } \Omega, \quad \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

(Pipher and Verchota, 1992) If $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $1 < p < 2 + \varepsilon$, then we can solve the problem

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \nabla u\big|_{\partial\Omega} = \dot{f}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)}.$$

(Shen, 2006) If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and A is constant, and $2 - \varepsilon < p < 2 + 4/\max(0, d-3) + \varepsilon$, then we can solve the problems

$$\nabla^m \cdot A \nabla^m u = 0 \text{ in } \Omega, \ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \ \|N(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}.$$

(Shen, 2006–7) If $1 + \max(0, d - 3 - \varepsilon)/(d+1) < p < 2 + \varepsilon$ then we can solve the problems

$$\nabla^m \cdot A \nabla^m u = 0 \text{ in } \Omega, \ \nabla^{m-1} u\big|_{\partial\Omega} = \dot{f}, \ \|N(\nabla^m u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot{f}\|_{L^p(\partial\Omega)},$$

$$\Delta^2 u = 0 \text{ in } \Omega, \quad \dot{M}_A^\Omega u = \dot{g}, \quad \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{L^p(\partial\Omega)}.$$

Our goal

Conjecture (B., Hofmann, Mayboroda)

Let Ω be the region above a Lipschitz graph. Let A be a self-adjoint, t-independent, bounded elliptic matrix of coefficients.
Then we can solve the Dirichlet problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \nabla^{m-1}u\big|_{\partial\Omega} = \dot{f}, \\ \|\widetilde{N}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{f}\|_{L^p(\partial\Omega)}, \end{cases} \qquad \begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \nabla^{m-1}u\big|_{\partial\Omega} = \dot{f}, \\ \|\widetilde{N}(\nabla^{m}u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau\dot{f}\|_{L^p(\partial\Omega)}, \end{cases}$$

and the Neumann problems

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \dot{M}_\Omega^A u = \dot{g}, \\ \|\widetilde{N}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{\dot{W}^p_{-1}(\partial\Omega)}, \end{cases} \qquad \begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega, \\ \dot{M}_\Omega^A u = \dot{g}, \\ \|\widetilde{N}(\nabla^{m}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot{g}\|_{L^p(\partial\Omega)}. \end{cases}$$

The Rellich identity

Theorem

If A is self-adjoint and t-independent, if $\Omega = \{(x,t) : t > \psi(x)\}$ is the domain above a Lipschitz graph, and if u satisfies appropriate bounds, then

$$\|\nabla^{m-1} u|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} \lesssim \|\dot M_A u\|_{L^2(\partial\Omega)}.$$

The Rellich identity

Theorem

If A is self-adjoint and t-independent, if $\Omega = \{(x,t) : t > \psi(x)\}$ is the domain above a Lipschitz graph, and if u satisfies appropriate bounds, then

$$\|\nabla^{m-1} u|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} \lesssim \|\dot M_A u\|_{L^2(\partial\Omega)}.$$

Proof.
$$\begin{aligned}\|\nabla^{m-1} u|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} &= \|\nabla_\tau \nabla^{m-1} u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \\ &\le \|\nabla^m u|_{\partial\Omega}\|_{L^2(\partial\Omega)},\end{aligned}$$

The Rellich identity

Theorem

If A is self-adjoint and t-independent, if $\Omega = \{(x,t) : t > \psi(x)\}$ is the domain above a Lipschitz graph, and if u satisfies appropriate bounds, then

$$\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} \lesssim \|\dot M_A u\|_{L^2(\partial\Omega)}.$$

Proof.
$$\begin{aligned}\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} &= \|\nabla_\tau \nabla^{m-1} u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)} \\ &\leq \|\nabla^m u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)}, \text{ and}\end{aligned}$$

$$2\,\mathrm{Re}\int_{\partial\Omega} \nabla^{m-1}\partial_t \bar u \cdot \dot M_A u \, d\sigma = 2\,\mathrm{Re}\int_\Omega \nabla^m \partial_t \bar u \cdot A\nabla^m u$$
$$= \int_\Omega \frac{\partial}{\partial t}(\nabla^m \bar u \cdot A \nabla^m u) = -\int_{\mathbb{R}^{d-1}} \overline{\nabla^m u(x,\psi(x))} \cdot A(x) \nabla^m u(x,\psi(x))\,dx.$$

The Rellich identity

Theorem

If A is self-adjoint and t-independent, if $\Omega = \{(x,t) : t > \psi(x)\}$ is the domain above a Lipschitz graph, and if u satisfies appropriate bounds, then

$$\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} \lesssim \|\dot M_A u\|_{L^2(\partial\Omega)}.$$

Proof.
$$\begin{aligned}\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} &= \|\nabla_\tau \nabla^{m-1} u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)} \\ &\leq \|\nabla^m u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)}, \text{ and}\end{aligned}$$

$$\begin{aligned}2\,\mathrm{Re}\int_{\partial\Omega} \nabla^{m-1}\partial_t \bar u \cdot \dot M_A u \, d\sigma &= 2\,\mathrm{Re}\int_\Omega \nabla^m \partial_t \bar u \cdot A\nabla^m u \\ = \int_\Omega \frac{\partial}{\partial t}(\nabla^m \bar u \cdot A \nabla^m u) &= -\int_{\mathbb{R}^{d-1}} \overline{\nabla^m u(x,\psi(x))} \cdot A(x) \nabla^m u(x,\psi(x))\, dx.\end{aligned}$$

So $\|\nabla^m u\big|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \lesssim \|\nabla^{m-1}\partial_t u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)} \|\dot M_A u\|_{L^2(\partial\Omega)}$.

The Rellich identity

Theorem

If A is self-adjoint and t-independent, if $\Omega = \{(x,t) : t > \psi(x)\}$ is the domain above a Lipschitz graph, and if u satisfies appropriate bounds, then

$$\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} \lesssim \|\dot M_A u\|_{L^2(\partial\Omega)}.$$

Proof. $\|\nabla^{m-1} u\big|_{\partial\Omega}\|_{\dot W^{1,2}(\partial\Omega)} = \|\nabla_\tau \nabla^{m-1} u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)}$
$\leq \|\nabla^m u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)}$, and

$$2\,\mathrm{Re}\int_{\partial\Omega} \nabla^{m-1}\partial_t \bar u \cdot \dot M_A u \, d\sigma = 2\,\mathrm{Re}\int_\Omega \nabla^m \partial_t \bar u \cdot A \nabla^m u$$
$$= \int_\Omega \frac{\partial}{\partial t}(\nabla^m \bar u \cdot A\nabla^m u) = -\int_{\mathbb{R}^{d-1}} \overline{\nabla^m u(x,\psi(x))} \cdot A(x) \nabla^m u(x,\psi(x))\, dx.$$

So $\|\nabla^m u\big|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \lesssim \|\nabla^{m-1}\partial_t u\big|_{\partial\Omega}\|_{L^2(\partial\Omega)} \|\dot M_A u\|_{L^2(\partial\Omega)}$.
In the case $m=1$, $M_A u = \nu \cdot A\nabla u$, so $\|M_A u\|_{L^2} \lesssim \|\nabla^1 u\big|_{\partial\Omega}\|_{L^2}$.
Careful algebra shows $\|M_A u\|_{L^2(\partial\Omega)} \lesssim \|u\big|_{\partial\Omega}\|_{\dot W^2_1(\partial\Omega)}$.

The Green's formula: harmonic functions

Let $E_X(Y) = \dfrac{c_d}{|X-Y|^{d-2}}$ (in $\mathbb{R}^d$) or $E_X(Y) = -\frac{1}{2\pi}\log|X-Y|$ (in $\mathbb{R}^2$).

Then

$$-\Delta E_X = \delta_X.$$

The Green's formula: harmonic functions

Let $E_X(Y) = \dfrac{c_d}{|X-Y|^{d-2}}$ (in $\mathbb{R}^d$) or $E_X(Y) = -\frac{1}{2\pi}\log|X-Y|$ (in $\mathbb{R}^2$).

Then

$$-\Delta E_X = \delta_X.$$

So if $\Delta u = 0$ in Ω,

$$\begin{aligned} u(X) &= \int_\Omega -\Delta E_X\, u \\ &= -\int_{\partial\Omega} \nu\cdot\nabla E_X\, u\, d\sigma + \int_{\partial\Omega} E_X\, \nu\cdot\nabla u\, d\sigma - \int_\Omega E_X\, \Delta u. \end{aligned}$$

The Green's formula: harmonic functions

Let $E_X(Y) = \dfrac{c_d}{|X-Y|^{d-2}}$ (in $\mathbb{R}^d$) or $E_X(Y) = -\frac{1}{2\pi}\log|X-Y|$ (in $\mathbb{R}^2$).

Then

$$-\Delta E_X = \delta_X.$$

So if $\Delta u = 0$ in Ω,

$$\begin{aligned} u(X) &= \int_\Omega -\Delta E_X\, u \\ &= -\int_{\partial\Omega} \nu\cdot\nabla E_X\, u\, d\sigma + \int_{\partial\Omega} E_X\, \nu\cdot\nabla u\, d\sigma - \int_\Omega E_X\, \Delta u. \end{aligned}$$

We define $\mathcal{D}_\Omega f(X) = \int_{\partial\Omega} \nu\cdot\nabla E_X\, f\, d\sigma$, $\mathcal{S}_\Omega g(X) = \int_{\partial\Omega} E_X\, g\, d\sigma$,

so if $\Delta u = 0$ in Ω then $u = -\mathcal{D}_\Omega(u\big|_{\partial\Omega}) + \mathcal{S}_\Omega(\nu\cdot\nabla u)$

The Green's formula: harmonic functions

Let $E_X(Y) = \dfrac{c_d}{|X-Y|^{d-2}}$ (in $\mathbb{R}^d$) or $E_X(Y) = -\frac{1}{2\pi}\log|X-Y|$ (in $\mathbb{R}^2$).

Then

$$-\Delta E_X = \delta_X.$$

So if $\Delta u = 0$ in Ω,

$$\begin{aligned} u(X) &= \int_\Omega -\Delta E_X\, u \\ &= -\int_{\partial\Omega} \nu\cdot\nabla E_X\, u\, d\sigma + \int_{\partial\Omega} E_X\, \nu\cdot\nabla u\, d\sigma - \int_\Omega E_X\, \Delta u. \end{aligned}$$

We define $\mathcal{D}_\Omega f(X) = \int_{\partial\Omega} \nu\cdot\nabla E_X\, f\, d\sigma$, $\mathcal{S}_\Omega g(X) = \int_{\partial\Omega} E_X\, g\, d\sigma$,

so if $\Delta u = 0$ in Ω then $u = -\mathcal{D}_\Omega(u\big|_{\partial\Omega}) + \mathcal{S}_\Omega(\nu\cdot\nabla u)$ and
$\|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|N(\nabla\mathcal{D}_\Omega(u\big|_{\partial\Omega}))\|_{L^p(\partial\Omega)} + \|N(\nabla\mathcal{S}_\Omega(\nu\cdot\nabla u))\|_{L^p(\partial\Omega)}$.

Layer potentials

We can generalize layer potentials so that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$u(X) = -\mathcal{D}^A(\nabla^{m-1} u\big|_{\partial\Omega})(X) + \mathcal{S}^A(\dot{M}_\Omega^A u)(X).$$

Layer potentials

We can generalize layer potentials so that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$u(X) = -\mathcal{D}^A(\nabla^{m-1} u\big|_{\partial\Omega})(X) + \mathcal{S}^A(\dot{M}_\Omega^A u)(X).$$

Therefore,

$$\begin{aligned}\|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \\ &\lesssim \|\widetilde{N}(\nabla^m \mathcal{D}^A \nabla^{m-1} u\big|_{\partial\Omega})\|_{L^2(\partial\Omega)} + \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{M}_\Omega^A u)\|_{L^2(\partial\Omega)}.\end{aligned}$$

Layer potentials

We can generalize layer potentials so that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$u(X) = -\mathcal{D}^A(\nabla^{m-1} u\big|_{\partial\Omega})(X) + \mathcal{S}^A(\dot{M}_\Omega^A u)(X).$$

Therefore,

$$\|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \lesssim \|\widetilde{N}(\nabla^m \mathcal{D}^A \nabla^{m-1} u\big|_{\partial\Omega})\|_{L^2(\partial\Omega)} + \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{M}_\Omega^A u)\|_{L^2(\partial\Omega)}.$$

If $\quad \|\widetilde{N}(\nabla^m \mathcal{D}^A \dot{f})\|_{L^2(\partial\Omega)} \lesssim \|\dot{f}\|_{\dot{W}_1^2(\partial\Omega)}, \quad \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{g})\|_{L^2(\partial\Omega)} \lesssim \|\dot{g}\|_{L^2(\partial\Omega)},$

Layer potentials

We can generalize layer potentials so that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$u(X) = -\mathcal{D}^A(\nabla^{m-1} u\big|_{\partial\Omega})(X) + \mathcal{S}^A(\dot{M}_\Omega^A u)(X).$$

Therefore,

$$\|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \lesssim \|\widetilde{N}(\nabla^m \mathcal{D}^A \nabla^{m-1} u\big|_{\partial\Omega})\|_{L^2(\partial\Omega)} + \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{M}_\Omega^A u)\|_{L^2(\partial\Omega)}.$$

If $\quad \|\widetilde{N}(\nabla^m \mathcal{D}^A \dot{f})\|_{L^2(\partial\Omega)} \lesssim \|\dot{f}\|_{\dot{W}_1^2(\partial\Omega)}, \quad \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{g})\|_{L^2(\partial\Omega)} \lesssim \|\dot{g}\|_{L^2(\partial\Omega)},$

then $\qquad \|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \lesssim \|\nabla^{m-1} u\|_{\dot{W}_1^2(\partial\Omega)} + \|\dot{M}_\Omega^A u\|_{L^2(\partial\Omega)}$

Layer potentials

We can generalize layer potentials so that if $\nabla^m \cdot A\nabla^m u = 0$ in Ω, then

$$u(X) = -\mathcal{D}^A(\nabla^{m-1} u|_{\partial\Omega})(X) + \mathcal{S}^A(\dot{M}_\Omega^A u)(X).$$

Therefore,

$$\begin{aligned}\|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \\ &\lesssim \|\widetilde{N}(\nabla^m \mathcal{D}^A \nabla^{m-1} u|_{\partial\Omega})\|_{L^2(\partial\Omega)} + \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{M}_\Omega^A u)\|_{L^2(\partial\Omega)}.\end{aligned}$$

If $\quad \|\widetilde{N}(\nabla^m \mathcal{D}^A \dot{f})\|_{L^2(\partial\Omega)} \lesssim \|\dot{f}\|_{\dot{W}_1^2(\partial\Omega)}, \quad \|\widetilde{N}(\nabla^m \mathcal{S}^A \dot{g})\|_{L^2(\partial\Omega)} \lesssim \|\dot{g}\|_{L^2(\partial\Omega)},$

then $\quad \|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \lesssim \|\nabla^{m-1} u\|_{\dot{W}_1^2(\partial\Omega)} + \|\dot{M}_\Omega^A u\|_{L^2(\partial\Omega)}$

and by the Rellich identity $\quad \|\widetilde{N}(\nabla^m u)\|_{L^2(\partial\Omega)} \lesssim \|\dot{M}_\Omega^A u\|_{L^2(\partial\Omega)}.$

Boundedness of layer potentials and trace theorems

Theorem (B., Hofmann, Mayboroda, 2017)

Suppose that A is elliptic and t-independent. Then we have the estimates

$$\int_{\mathbb{R}^d_+} |\nabla^m \partial_t \mathcal{S}^A \dot{g}(x,t)|^2 \, t \, dx \, dt \lesssim \|\dot{g}\|^2_{L^2(\mathbb{R}^{d-1})},$$

$$\int_{\mathbb{R}^d_+} |\nabla^m \partial_t \mathcal{D}^A \dot{f}(x,t)|^2 \, t \, dx \, dt \lesssim \|\nabla_\tau \dot{f}\|^2_{L^2(\mathbb{R}^{d-1})} = \|\dot{f}\|^2_{\dot{W}^2_1(\mathbb{R}^{d-1})}.$$

Boundedness of layer potentials and trace theorems

Theorem (B., Hofmann, Mayboroda, 2017)

Suppose that A is elliptic and t-independent. Then we have the estimates

$$\int_{\mathbb{R}^d_+} |\nabla^m \partial_t \mathcal{S}^A \dot{g}(x,t)|^2 \, t \, dx \, dt \lesssim \|\dot{g}\|^2_{L^2(\mathbb{R}^{d-1})},$$

$$\int_{\mathbb{R}^d_+} |\nabla^m \partial_t \mathcal{D}^A \dot{f}(x,t)|^2 \, t \, dx \, dt \lesssim \|\nabla_\tau \dot{f}\|^2_{L^2(\mathbb{R}^{d-1})} = \|\dot{f}\|^2_{\dot{W}^2_1(\mathbb{R}^{d-1})}.$$

Theorem (B., Hofmann, Mayboroda)

Suppose that A is elliptic and t-independent. Then we have the estimates

$$\int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m \mathcal{S}^A \dot{g})(x)^2 \, dx \lesssim \|\dot{g}\|^2_{L^2(\mathbb{R}^{d-1})},$$

$$\int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m \mathcal{D}^A \dot{f})(x)^2 \, dx \lesssim \|\nabla_\tau \dot{f}\|^2_{L^2(\mathbb{R}^{d-1})} = \|\dot{f}\|^2_{\dot{W}^2_1(\mathbb{R}^{d-1})}.$$

The Neumann problem

Theorem (B., Hofmann, Mayboroda)

Let A be a self-adjoint, t-independent, elliptic matrix of coefficients. Then there is a solution to the L^2-Neumann problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \dot{M}_A u = \dot{g} \text{ on } \partial\mathbb{R}^d_+, \\ \int_{\mathbb{R}^d_+} |\nabla^m \partial_t u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m u)(x)^2 \, dx \lesssim \|\dot{g}\|^2_{L^2(\partial\mathbb{R}^d_+)} \end{cases}$$

that is unique up to adding polynomials of degree $m-1$.

Harmonic layer potentials

Recall: $E_X(Y) = \dfrac{c_d}{|X-Y|^{d-2}}$ in $\mathbb{R}^d$ and $E_X(Y) = -\frac{1}{2\pi}\log|X-Y|$ in $\mathbb{R}^2$.
Formally

$$-\Delta E_X = \delta_X.$$

So if $\Delta u = 0$ in Ω,

$$\begin{aligned} u(X) &= \int_\Omega -\Delta E_X \, u \\ &= -\int_{\partial\Omega} \nu\cdot\nabla E_X \, u \, d\sigma + \int_{\partial\Omega} E_X \, \nu\cdot\nabla u \, d\sigma - \int_\Omega E_X \, \Delta u. \end{aligned}$$

We define $\mathcal{D}_\Omega f(X) = \int_{\partial\Omega} \nu\cdot\nabla E_X \, f \, d\sigma$, $\mathcal{S}_\Omega g(X) = \int_{\partial\Omega} E_X \, g \, d\sigma$,

so if $\Delta u = 0$ in Ω then $u = -\mathcal{D}_\Omega(u\big|_{\partial\Omega}) + \mathcal{S}_\Omega(\nu\cdot\nabla u)$.

Layer potentials and well posedness: C^1 domains

Theorem (Fabes, Jodeit, Rivière, 1978)

Let $\Omega = \Omega_+$ be a bounded C^1 domain, and let $\partial\Omega_+ = \partial\Omega_-$, $\Omega_+ \cap \Omega_- = \emptyset$. Then we have the bounds

$$\|N(\mathcal{D}_\Omega \varphi)\|_{L^p(\partial\Omega)} \lesssim \|\varphi\|_{L^p(\partial\Omega)}, \quad \|N(\nabla \mathcal{D}_\Omega \varphi)\|_{L^p(\partial\Omega)} \lesssim \|\varphi\|_{\dot W^{1,p}(\partial\Omega)},$$
$$\|N(\nabla \mathcal{S}_\Omega \gamma)\|_{L^p(\partial\Omega)} \lesssim \|\gamma\|_{L^p(\partial\Omega)}$$

and the formulas

$$\mathcal{D}_\Omega \varphi\big|_{\partial\Omega_\pm} = \mp\frac{1}{2}\varphi + K\varphi, \quad \nu_\pm \cdot \nabla \mathcal{S}_\Omega \gamma\big|_{\partial\Omega_\pm} = \frac{1}{2}\gamma \pm K^*\gamma$$

where K is compact on $L^p(\partial\Omega)$ and $\dot W^{1,p}(\partial\Omega)$, $1 < p < \infty$.

Corollary Let $f \in L^p(\partial\Omega)$. Then there is some $\varphi \in L^p(\partial\Omega)$ such that $u = \mathcal{D}_\Omega \varphi$ satisfies

$$\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Layer potentials and well posedness: Lipschitz domains

Let Ω be a bounded simply connected Lipschitz domain.
(Dahlberg, 1977 and 1979) If $2-\varepsilon < p < \infty$, then we can solve

$$\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}. \tag{1}$$

(Jerison and Kenig, 1981) We can solve

$$\begin{gathered}\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|N(\nabla u)\|_{L^2(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,2}(\partial\Omega)}, \\ \Delta u = 0 \text{ in } \Omega, \quad \nu\cdot\nabla u\big|_{\partial\Omega} = g, \quad \|N(\nabla u)\|_{L^2(\partial\Omega)} \lesssim \|g\|_{L^2(\partial\Omega)}.\end{gathered} \tag{2}$$

(Verchota, 1984) If $1 < p < 2+\varepsilon$, then we can solve

$$\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)} \tag{3}$$

and the solutions u to the problems (1), (2) and (3) may be written as layer potentials.

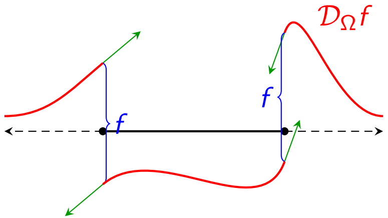
Jump relations

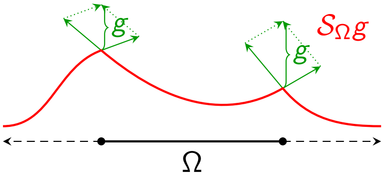
If $\Omega = \Omega_+ = \mathbb{R}^d \setminus \overline{\Omega_-}$ is a bounded Lipschitz domain, and if f and g are continuous on $\partial\Omega$, then

$$\Delta(\mathcal{D}_\Omega f) = 0 \text{ and } \Delta(\mathcal{S}_\Omega g) = 0 \text{ in } \mathbb{R}^d \setminus \partial\Omega,$$

$\mathcal{D}_\Omega f$ and $\nabla\mathcal{S}_\Omega g$ extend to functions continuous on $\overline{\Omega_+}$ and $\overline{\Omega_-}$, and

$$\mathcal{D}_\Omega f\big|_{\partial\Omega_+} - \mathcal{D}_\Omega f\big|_{\partial\Omega_+} = -f, \quad \nu_+ \cdot \nabla\mathcal{D}_\Omega f\big|_{\partial\Omega_+} + \nu_- \cdot \nabla\mathcal{D}_\Omega f\big|_{\partial\Omega_-} = 0,$$
$$\mathcal{S}_\Omega g\big|_{\partial\Omega_+} - \mathcal{S}_\Omega g\big|_{\partial\Omega_+} = 0, \qquad \nu_+ \cdot \nabla\mathcal{S}_\Omega g\big|_{\partial\Omega_+} + \nu_- \cdot \nabla\mathcal{S}_\Omega g\big|_{\partial\Omega_-} = g.$$





Well posedness and invertibility

Consider the Dirichlet regularity problem

$$\Delta u = 0 \text{ in } \Omega_+, \quad u\big|_{\partial\Omega_+} = f, \quad \|N_+(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}$$

Suppose that $\|N_\pm(\nabla \mathcal{S}_\Omega g)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

Well posedness and invertibility

Consider the Dirichlet regularity problem

$$\Delta u = 0 \text{ in } \Omega_+, \quad u\big|_{\partial\Omega_+} = f, \quad \|N_+(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}$$

Suppose that $\|N_\pm(\nabla \mathcal{S}_\Omega g)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto $L^p(\partial\Omega) \mapsto \dot W^{1,p}(\partial\Omega)$ with a bounded right inverse:
If $f \in \dot W^{1,p}(\partial\Omega)$ then $f = \mathcal{S}_\Omega g\big|_{\partial\Omega}$ for some g, $\|g\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}$.
Then there is at least one solution to the regularity problem

Well posedness and invertibility

Consider the Dirichlet regularity problems

$$\Delta u = 0 \text{ in } \Omega_+, \quad u\big|_{\partial\Omega_+} = f, \quad \|N_+(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)},$$
$$\Delta u = 0 \text{ in } \Omega_-, \quad u\big|_{\partial\Omega_-} = f, \quad \|N_-(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}.$$

Suppose that $\|N_\pm(\nabla \mathcal{S}_\Omega g)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto $L^p(\partial\Omega) \mapsto \dot W^{1,p}(\partial\Omega)$ with a bounded right inverse:
If $f \in \dot W^{1,p}(\partial\Omega)$ then $f = \mathcal{S}_\Omega g\big|_{\partial\Omega}$ for some g, $\|g\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}$.
Then there is at least one solution to each of the regularity problems.

(Verchota, 1984) Suppose that we have uniqueness of solutions.
Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one $L^p(\partial\Omega) \mapsto \dot W^{1,p}(\partial\Omega)$ with bounded left inverse:
$\|g\|_{L^p(\partial\Omega)} \lesssim \|\mathcal{S}_\Omega g\big|_{\partial\Omega}\|_{\dot W^{1,p}(\partial\Omega)}$ for all $g \in L^p(\partial\Omega)$.

Well posedness and invertibility

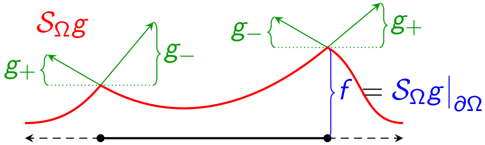
Consider the Dirichlet regularity problems

$$\Delta u = 0 \text{ in } \Omega_+, \quad u\big|_{\partial\Omega_+} = f, \quad \|N_+(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)},$$
$$\Delta u = 0 \text{ in } \Omega_-, \quad u\big|_{\partial\Omega_-} = f, \quad \|N_-(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}.$$

Suppose that $\|N_\pm(\nabla \mathcal{S}_\Omega g)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto $L^p(\partial\Omega) \mapsto \dot W^{1,p}(\partial\Omega)$ with a bounded right inverse:
If $f \in \dot W^{1,p}(\partial\Omega)$ then $f = \mathcal{S}_\Omega g\big|_{\partial\Omega}$ for some g, $\|g\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot W^{1,p}(\partial\Omega)}$.
Then there is at least one solution to each of the regularity problems.

(Verchota, 1984) Suppose that we have uniqueness of solutions.
Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one $L^p(\partial\Omega) \mapsto \dot W^{1,p}(\partial\Omega)$ with bounded left inverse:
$\|g\|_{L^p(\partial\Omega)} \lesssim \|\mathcal{S}_\Omega g\big|_{\partial\Omega}\|_{\dot W^{1,p}(\partial\Omega)}$ for all $g \in L^p(\partial\Omega)$.



Well posedness and invertibility

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto with a bounded right inverse. Then we have existence of solutions.

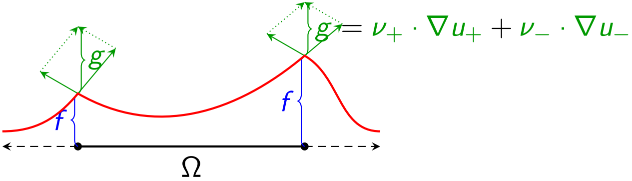
(Verchota, 1984) Suppose that we have uniqueness of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one with bounded left inverse.

Well posedness and invertibility

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto with a bounded right inverse. Then we have existence of solutions.

(Verchota, 1984) Suppose that we have uniqueness of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one with bounded left inverse.

(B., Mayboroda, 2013) Suppose that we have existence of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is onto with bounded right inverse.

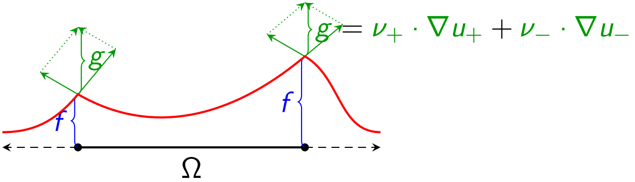


Well posedness and invertibility

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto with a bounded right inverse. Then we have existence of solutions.

(Verchota, 1984) Suppose that we have uniqueness of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one with bounded left inverse.

(B., Mayboroda, 2013) Suppose that we have existence of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is onto with bounded right inverse.



(B., Mayboroda, 2016) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is one-to-one with a bounded left inverse. Then there is at most one solution:

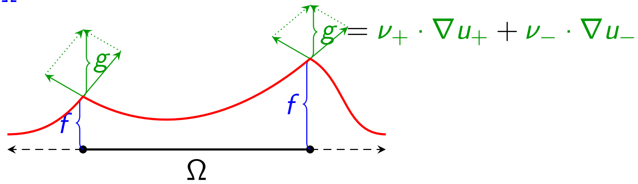
$$u \quad + \mathcal{D}_\Omega(u\big|_{\partial\Omega}) \quad = \mathcal{S}_\Omega(\nu \cdot \nabla u)$$

Well posedness and invertibility

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto with a bounded right inverse. Then we have existence of solutions.

(Verchota, 1984) Suppose that we have uniqueness of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one with bounded left inverse.

(B., Mayboroda, 2013) Suppose that we have existence of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is onto with bounded right inverse.



(B., Mayboroda, 2016) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is one-to-one with a bounded left inverse. Then there is at most one solution:

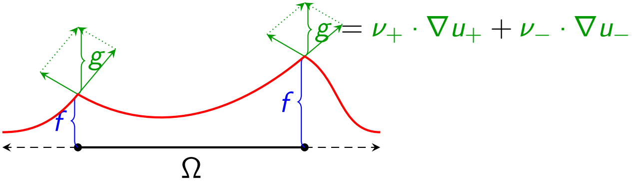
$$u\big|_{\partial\Omega} + \mathcal{D}_\Omega(u\big|_{\partial\Omega})\big|_{\partial\Omega} = \mathcal{S}_\Omega(\nu \cdot \nabla u)\big|_{\partial\Omega}$$

Well posedness and invertibility

(The classic method of layer potentials) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is onto with a bounded right inverse. Then we have existence of solutions.

(Verchota, 1984) Suppose that we have uniqueness of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is one-to-one with bounded left inverse.

(B., Mayboroda, 2013) Suppose that we have existence of solutions. Then $\mathcal{S}_\Omega\big|_{\partial\Omega}$ is onto with bounded right inverse.



(B., Mayboroda, 2016) Suppose that $g \mapsto \mathcal{S}_\Omega g\big|_{\partial\Omega}$ is one-to-one with a bounded left inverse. Then there is at most one solution:

$$u\big|_{\partial\Omega} + \mathcal{D}_\Omega(u\big|_{\partial\Omega})\big|_{\partial\Omega} = \mathcal{S}_\Omega(\nu \cdot \nabla u)\big|_{\partial\Omega}$$

$$\text{so } u = -\mathcal{D}_\Omega(u\big|_{\partial\Omega}) + \mathcal{S}_\Omega\big((\mathcal{S}_\Omega\big|_{\partial\Omega})^{-1}(u\big|_{\partial\Omega} + \mathcal{D}_\Omega(u\big|_{\partial\Omega})\big|_{\partial\Omega})\big)$$

The Green's formula: second-order operators

If A is real (Grüter, Widman, 1982; Kenig, Ni, 1985), complex and satisfies the De Giorgi-Nash-Moser condition (Hofmann, Kim, 2007), or satisfies the Moser condition (Rosén, 2013) then there is a fundamental solution $E_X^A(Y)$ such that

$$-\nabla \cdot A^T \nabla E_X^A = \delta_X.$$

Then formally

$$u(X) = -\int_{\partial\Omega} \nu \cdot A^T \nabla E_X^A \, u \, d\sigma + \int_{\partial\Omega} E_X^A \, \nu \cdot A\nabla u \, d\sigma - \int_{\Omega} E_X^A \, \nabla \cdot A\nabla u.$$

In particular, if $\nabla \cdot A\nabla u = 0$ then we expect that

$$u(X) = -\int_{\partial\Omega} \nu \cdot A^T \nabla E_X^A \, u \, d\sigma + \int_{\partial\Omega} E_X^A \, \nu \cdot A\nabla u \, d\sigma.$$

The Green's formula: second-order operators

If A is real (Grüter, Widman, 1982; Kenig, Ni, 1985), complex and satisfies the De Giorgi-Nash-Moser condition (Hofmann, Kim, 2007), or satisfies the Moser condition (Rosén, 2013) then there is a fundamental solution $E_X^A(Y)$ such that

$$-\nabla \cdot A^T \nabla E_X^A = \delta_X.$$

Then formally

$$u(X) = -\int_{\partial\Omega} \nu \cdot A^T \nabla E_X^A \, u \, d\sigma + \int_{\partial\Omega} E_X^A \, \nu \cdot A\nabla u \, d\sigma - \int_\Omega E_X^A \, \nabla \cdot A\nabla u.$$

In particular, if $\nabla \cdot A\nabla u = 0$ then we expect that

$$u(X) = -\int_{\partial\Omega} \nu \cdot A^T \nabla E_X^A \, u \, d\sigma + \int_{\partial\Omega} E_X^A \, \nu \cdot A\nabla u \, d\sigma.$$

It takes quite a bit of work to show that this is actually true! (Kenig, Rule, 2009; Alfonseca, Auscher, Axelsson, Hofmann, Kim, 2011; B., Mayboroda, 2013, 2016; Auscher, Mourgoglou, 2014; Hofmann, Kenig, Mayboroda, Pipher, 2015; Hofmann, Mayboroda, Mourgoglou, 2015; Hofmann, Mitrea, Morris, 2015; others)

Higher order layer potentials

If A has constant coefficients, we can construct the fundamental solution E_X^A to

$$(-1)^m \nabla^m \cdot A^T \nabla^m E_X^A = \delta_X$$

using the Fourier transform.
We cound define

$$\mathcal{D}_\Omega^A \dot{f}(X) = \int_{\partial\Omega} \dot{M}_\Omega^{A^T} E_X^A \cdot \dot{f}\, d\sigma, \quad \mathcal{S}_\Omega^A \dot{g}(X) = \int_{\partial\Omega} \nabla^{m-1} E_X^A \cdot \dot{g}\, d\sigma.$$

(Cohen, Gosselin, 1983/1985; Verchota, 2005; I. Mitrea, M. Mitrea, 2013)

Higher order layer potentials

If A has constant coefficients, we can construct the fundamental solution E_X^A to

$$(-1)^m \nabla^m \cdot A^T \nabla^m E_X^A = \delta_X$$

using the Fourier transform.
We cound define

$$\mathcal{D}_\Omega^A \dot{f}(X) = \int_{\partial\Omega} \dot{M}_\Omega^{A^T} E_X^A \cdot \dot{f}\, d\sigma, \quad \mathcal{S}_\Omega^A \dot{g}(X) = \int_{\partial\Omega} \nabla^{m-1} E_X^A \cdot \dot{g}\, d\sigma.$$

(Cohen, Gosselin, 1983/1985; Verchota, 2005; I. Mitrea, M. Mitrea, 2013)

(B., 2016) There is a fundamental solution even for variable coefficients. . .

The fundamental solution

(Hofmann and Kim, 2007; B., 2016) The (gradient of the) fundamental solution $\nabla_Y E^A_X(Y)$ is the kernel of the operator $\Pi^{A^T} = (-1)^m (L^T)^{-1} \nabla^m$

The fundamental solution

(Hofmann and Kim, 2007; B., 2016) The (gradient of the) fundamental solution $\nabla_Y E^A_X(Y)$ is the kernel of the operator $\Pi^{A^T} = (-1)^m (L^T)^{-1} \nabla^m$ where $\Pi^A \dot{H}$ satisfies

$$\int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \Pi^A \dot{H} = \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot \dot{H}$$

for all $\varphi \in \dot{W}^{m,2}(\mathbb{R}^d)$.

The fundamental solution

(Hofmann and Kim, 2007; B., 2016) The (gradient of the) fundamental solution $\nabla_Y E_X^A(Y)$ is the kernel of the operator $\Pi^{A^T} = (-1)^m (L^T)^{-1} \nabla^m$ where $\Pi^A \dot{H}$ satisfies

$$\int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \Pi^A \dot{H} = \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot \dot{H}$$

for all $\varphi \in \dot{W}^{m,2}(\mathbb{R}^d)$.

Theorem (Lax-Milgram)

Let H be a Hilbert space. Let $B : H \times H \mapsto \mathbb{C}$ and suppose:

- *B is bilinear,*
- $|B(v,w)| \leq \Lambda \|v\| \, \|w\|$,
- $|B(v,v)| \geq \lambda \|v\|^2$.

If $T : H \mapsto \mathbb{C}$ is a bounded linear operator, then there exists a unique element $u_T \in H$ such that $\overline{T(v)} = B(v, u_T)$, and $\|u_T\|_H \leq \frac{1}{\lambda} \|T\|_{H \mapsto \mathbb{C}}$.

Another way to write jump relations

Let f, g be nice functions defined on $\partial\Omega$. Recall that $\mathcal{D}_\Omega f$, $\mathcal{S}_\Omega g$ satisfy:

$$\begin{aligned} \Delta(\mathcal{S}_\Omega g) &= 0 \text{ in } \Omega_\pm, \\ \nu_+ \cdot \nabla \mathcal{S}_\Omega g + \nu_- \cdot \nabla \mathcal{S}_\Omega g &= g, \\ \mathcal{S}_\Omega g\big|_{\partial\Omega_+} &= \mathcal{S}_\Omega g\big|_{\partial\Omega_-}, \end{aligned}$$

$$\begin{aligned} \Delta(\mathcal{D}_\Omega f) &= 0 \text{ in } \Omega_\pm, \\ \nu_+ \cdot \nabla \mathcal{D}_\Omega f + \nu_- \cdot \nabla \mathcal{D}_\Omega f &= 0, \\ \mathcal{D}_\Omega f\big|_{\partial\Omega_+} &= \mathcal{D}_\Omega f\big|_{\partial\Omega_-} - f. \end{aligned}$$

Another way to write jump relations

Let f, g be nice functions defined on $\partial\Omega$. Recall that $\mathcal{D}_\Omega f$, $\mathcal{S}_\Omega g$ satisfy:

$$\begin{aligned} \Delta(\mathcal{S}_\Omega g) &= 0 \text{ in } \Omega_\pm, & \Delta(\mathcal{D}_\Omega f) &= 0 \text{ in } \Omega_\pm, \\ \nu_+ \cdot \nabla \mathcal{S}_\Omega g + \nu_- \cdot \nabla \mathcal{S}_\Omega g &= g, & \nu_+ \cdot \nabla \mathcal{D}_\Omega f + \nu_- \cdot \nabla \mathcal{D}_\Omega f &= 0, \\ \mathcal{S}_\Omega g\big|_{\partial\Omega_+} &= \mathcal{S}_\Omega g\big|_{\partial\Omega_-}, & \mathcal{D}_\Omega f\big|_{\partial\Omega_+} &= \mathcal{D}_\Omega f\big|_{\partial\Omega_-} - f. \end{aligned}$$

This means that

$$\int_{\partial\Omega} \operatorname{Tr}\varphi\, g\, d\sigma = \int_{\Omega_+} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g + \int_{\Omega_-} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g,$$

$$0 = \int_{\Omega_+} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f + \int_{\Omega_-} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f$$

and

$$\mathcal{S}_\Omega g \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d), \quad \mathcal{D}_\Omega f = v - \mathbf{1}_\Omega F \text{ where } F,\ v \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d),\ F\big|_{\partial\Omega} = f.$$

Harmonic layer potentials via the Riesz theorem

Let f, g be nice functions defined on $\partial\Omega$. Recall that

$$\int_{\partial\Omega} \operatorname{Tr}\varphi\, g\, d\sigma = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g, \qquad 0 = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f$$

and

$$\mathcal{S}_\Omega g \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d), \quad \mathcal{D}_\Omega f = v - \mathbf{1}_\Omega F \text{ where } F,\, v \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d),\ F\big|_{\partial\Omega} = f$$

Harmonic layer potentials via the Riesz theorem

Let f, g be nice functions defined on $\partial\Omega$. Recall that

$$\int_{\partial\Omega} \operatorname{Tr}\varphi\, g\, d\sigma = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g, \qquad 0 = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f$$

and

$$\mathcal{S}_\Omega g \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d), \quad \mathcal{D}_\Omega f = v - \mathbf{1}_\Omega F \text{ where } F,\ v \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d),\ F\big|_{\partial\Omega} = f$$

so $\int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla v = \int_\Omega \nabla\varphi \cdot \nabla F$.

Harmonic layer potentials via the Riesz theorem

Let f, g be nice functions defined on $\partial\Omega$. Recall that

$$\int_{\partial\Omega} \operatorname{Tr} \varphi \, g \, d\sigma = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{S}_\Omega g, \qquad 0 = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \mathcal{D}_\Omega f$$

and

$$\mathcal{S}_\Omega g \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d), \quad \mathcal{D}_\Omega f = v - \mathbf{1}_\Omega F \text{ where } F,\, v \in \dot{W}^{1,1}_{loc}(\mathbb{R}^d),\ F\big|_{\partial\Omega} = f$$

so $\int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla v = \int_\Omega \nabla\varphi \cdot \nabla F$.

It is well known that if Ω is a Lipschitz domain then

$$\{\operatorname{Tr} F : F \in \dot{W}^{1,2}(\mathbb{R}^d)\} = \dot{W}^{1/2,2}(\partial\Omega), \quad (\dot{W}^{1/2,2}(\partial\Omega))^* = \dot{W}^{-1/2,2}(\partial\Omega),$$

Harmonic layer potentials via the Riesz theorem

Let f, g be nice functions defined on $\partial\Omega$. Recall that

$$\int_{\partial\Omega} \operatorname{Tr}\varphi\, g\, d\sigma = \int_{\mathbb{R}^d} \nabla\varphi\cdot\nabla\mathcal{S}_\Omega g, \qquad 0 = \int_{\mathbb{R}^d} \nabla\varphi\cdot\nabla\mathcal{D}_\Omega f$$

and

$$\mathcal{S}_\Omega g \in \dot W^{1,1}_{loc}(\mathbb{R}^d), \quad \mathcal{D}_\Omega f = v - \mathbf{1}_\Omega F \text{ where } F,\, v\in \dot W^{1,1}_{loc}(\mathbb{R}^d),\ F\big|_{\partial\Omega} = f$$

so $\int_{\mathbb{R}^d}\nabla\varphi\cdot\nabla v = \int_\Omega \nabla\varphi\cdot\nabla F$.

It is well known that if Ω is a Lipschitz domain then

$$\{\operatorname{Tr} F : F\in \dot W^{1,2}(\mathbb{R}^d)\} = \dot W^{1/2,2}(\partial\Omega), \quad (\dot W^{1/2,2}(\partial\Omega))^* = \dot W^{-1/2,2}(\partial\Omega),$$

so if $f\in \dot W^{1/2,2}(\partial\Omega)$ and $g\in \dot W^{-1/2,2}(\partial\Omega)$, we can construct $v\in \dot W^{1,2}(\mathbb{R}^d)$ and $\mathcal{S}_\Omega g\in \dot W^{1,2}(\mathbb{R}^d)$ via the Riesz representation theorem.

General single layer potentials via the Lax-Milgram theorem

Let A be bounded and satisfy

$$\Re \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \varphi \geq \lambda \int_{\mathbb{R}^d} |\nabla^m \varphi|^2 \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

General single layer potentials via the Lax-Milgram theorem

Let A be bounded and satisfy

$$\Re \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \varphi \geq \lambda \int_{\mathbb{R}^d} |\nabla^m \varphi|^2 \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

If Ω is a Lipschitz domain, then the boundary trace operator $\operatorname{Tr} \nabla^{m-1}$ is bounded $\dot{W}^{m,2}(\mathbb{R}^d) \mapsto \dot{W}^{1/2,2}(\partial\Omega)$.

General single layer potentials via the Lax-Milgram theorem

Let A be bounded and satisfy

$$\Re \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \varphi \geq \lambda \int_{\mathbb{R}^d} |\nabla^m \varphi|^2 \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

If Ω is a Lipschitz domain, then the boundary trace operator $\operatorname{Tr} \nabla^{m-1}$ is bounded $\dot{W}^{m,2}(\mathbb{R}^d) \mapsto \dot{W}^{1/2,2}(\partial\Omega)$.

By the Lax-Milgram theorem, if $\dot{g} \in \dot{W}^{-1/2,2}(\partial\Omega)$, then there is a unique function $\mathcal{S}^L_\Omega \dot{g} \in \dot{W}^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m \mathcal{S}^L_\Omega \dot{g} = \int_{\partial\Omega} \operatorname{Tr} \nabla^{m-1} \varphi \cdot \dot{g} \, d\sigma \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

General single layer potentials via the Lax-Milgram theorem

Let A be bounded and satisfy

$$\Re \int_{\mathbb{R}^d} \overline{\nabla^m \varphi} \cdot A \nabla^m \varphi \geq \lambda \int_{\mathbb{R}^d} |\nabla^m \varphi|^2 \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

If Ω is a Lipschitz domain, then the boundary trace operator $\operatorname{Tr} \nabla^{m-1}$ is bounded $\dot{W}^{m,2}(\mathbb{R}^d) \mapsto \dot{W}^{1/2,2}(\partial\Omega)$.

By the Lax-Milgram theorem, if $\dot{g} \in \dot{W}^{-1/2,2}(\partial\Omega)$, then there is a unique function $\mathcal{S}^L_\Omega \dot{g} \in \dot{W}^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m \mathcal{S}^L_\Omega \dot{g} = \int_{\partial\Omega} \operatorname{Tr} \nabla^{m-1} \varphi \cdot \dot{g} \, d\sigma \quad \text{for all } \varphi \in \dot{W}^{1,2}(\mathbb{R}^d).$$

Then $L(\mathcal{S}^L_\Omega \dot{g}) = 0$ in Ω_+ and Ω_-, $\operatorname{Tr}_+ \nabla^{m-1} \mathcal{S}^L_\Omega \dot{g} = \operatorname{Tr}_- \nabla^{m-1} \mathcal{S}^L_\Omega \dot{g}$, and

$$\langle \nabla^{m-1} \varphi, \dot{\mathrm{M}}^+_A \mathcal{S}^L_\Omega \dot{g} + \dot{\mathrm{M}}^-_A \mathcal{S}^L_\Omega \dot{g} \rangle_{\partial\Omega} = \int_{\Omega_+} \overline{\nabla^m \varphi} \cdot A \nabla^m \mathcal{S}^L_\Omega \dot{g} + \int_{\Omega_-} \overline{\nabla^m \varphi} \cdot A \nabla^m \mathcal{S}^L_\Omega \dot{g}.$$

General double layer potentials via the Lax-Milgram theorem

Let $\dot{f} = \operatorname{Tr} \nabla^{m-1} F$ for some $F \in \dot{W}^{m,2}(\Omega)$. Let $\mathcal{D}^A_\Omega \dot{f}$ satisfy $(\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega F) \in \dot{W}^{m,2}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m (\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega F) = \int_\Omega \nabla^m \varphi \cdot A \nabla^m F.$$

General double layer potentials via the Lax-Milgram theorem

Let $\dot{f} = \operatorname{Tr} \nabla^{m-1} F$ for some $F \in \dot{W}^{m,2}(\Omega)$. Let $\mathcal{D}^A_\Omega \dot{f}$ satisfy $(\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega F) \in \dot{W}^{m,2}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m (\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega F) = \int_\Omega \nabla^m \varphi \cdot A \nabla^m F.$$

$\mathcal{D}^A_\Omega$ is well defined. If $\operatorname{Tr} \nabla^{m-1} F = \operatorname{Tr} \nabla^{m-1} \widetilde{F}$, then

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m (\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega \widetilde{F}) = \int_\Omega \nabla^m \varphi \cdot A \nabla^m \widetilde{F}$$

and $(\mathbf{1}_\Omega F - \mathbf{1}_\Omega \widetilde{F}) \in \dot{W}^{m,2}(\mathbb{R}^d)$:

$$\mathbf{1}_\Omega F - \mathbf{1}_\Omega \widetilde{F} = \begin{cases} F - \widetilde{F} & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

So $(\mathcal{D}^A_\Omega \dot{f} + \mathbf{1}_\Omega \widetilde{F}) \in \dot{W}^{m,2}(\Omega)$.

Properties of layer potentials

We have constructed layer potentials via the Lax-Milgram theorem.
Let $\dot{g} \in \dot{W}^{-1/2,2}(\partial\Omega)$, $\dot{f} \in \dot{W}A^2_{m-1,1/2}(\partial\Omega) \subsetneq \dot{W}^{1/2,2}(\partial\Omega)$.

- The conditions $L(\mathcal{D}^A_\Omega \dot{f}) = 0$, $L(\mathcal{S}^L_\Omega \dot{g}) = 0$ and the jump relations follow from the definition.
- Let $Lu = 0$ in Ω. Then $\mathcal{D}^A_\Omega(\operatorname{Tr} \nabla^{m-1} u) = -\mathbf{1}_\Omega u + v$, where

$$\int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m v = \int_\Omega \nabla^m \varphi \cdot A \nabla^m u = \int_{\partial\Omega} \operatorname{Tr} \nabla^{m-1} \varphi \cdot \dot{M}^\Omega_A u \, d\sigma$$
$$= \int_{\mathbb{R}^d} \nabla^m \varphi \cdot A \nabla^m \mathcal{S}^L_\Omega(\dot{M}^\Omega_A u)$$

  so we have the Green's formula $\mathbf{1}_\Omega u = -\mathcal{D}^A_\Omega(\operatorname{Tr} \nabla^{m-1} u) + \mathcal{S}^L_\Omega(\dot{M}^\Omega_A u)$.
- Boundary value problems are well posed if and only if boundary values of layer potentials are invertible.
- We can derive the formulas involving E^L_X using the connection between E^L_X and L^{-1}.
- We can derive adjoint relations: $(\dot{M}^\Omega_A \mathcal{D}^A_\Omega)^* = \dot{M}^\Omega_{A^*} \mathcal{D}^{A^*}_\Omega$, $(\operatorname{Tr} \nabla^{m-1} \mathcal{S}^L)^* = \operatorname{Tr} \nabla^{m-1} \mathcal{S}^{L^*}$, $(\operatorname{Tr}_+ \nabla^{m-1} \mathcal{D}^A_\Omega)^* = -\dot{M}^-_{A^*} \mathcal{S}^{L^*}$

The Neumann subregularity problem

Theorem (B., Hofmann, Mayboroda)

Let A be a self-adjoint, t-independent, elliptic matrix of coefficients. Then there is a solution to the $\dot{W}^{-1,2}$-Neumann problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \dot{M}_A u = \dot{g} \text{ on } \partial\mathbb{R}^d_+, \\ \int_{\mathbb{R}^d_+} |\nabla^m u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^{m-1} u)(x)^2 \, dx \lesssim \|\dot{g}\|^2_{\dot{W}^2_{-1}(\partial\mathbb{R}^d_+)} \end{cases}$$

that is unique up to adding polynomials of degree $m-1$.

The L^p problems

Recall:

(Pipher and Verchota, 1995) If Ω is a bounded Lipschitz domain and A is constant, and if $2-\varepsilon < p < 2+\varepsilon$, then we can solve the problems

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega,\ \ \nabla^{m-1}u\big|_{\partial\Omega} = \dot f,\ \ \|N(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \lesssim \|\dot f\|_{L^p(\partial\Omega)},$$

$$\nabla^m \cdot A\nabla^m u = 0 \text{ in } \Omega,\ \ \nabla^{m-1}u\big|_{\partial\Omega} = \dot f,\ \ \|N(\nabla^{m}u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_\tau \dot f\|_{L^p(\partial\Omega)}.$$

(Shen, 2006) For constant coefficient operators, using well posedness of the Dirichlet problem with $L^2(\partial\Omega)$ and $\dot W^{1,2}(\partial\Omega)$ boundary data, we can establish well posedness of the Dirichlet problem with boundary data in $L^p(\partial\Omega)$, $2 < p < 2 + \frac{4}{\max(0,d-3)} + \varepsilon$.

By duality we can establish well posedness for boundary data in $\dot W^{1,p}(\partial\Omega)$, $1 + \max(0, \frac{d-3}{d+1} - \varepsilon) < p < 2$.

(Shen, 2007) Similarly, we can solve the biharmonic Neumann problem with boundary data in $\dot W^{-1,p}(\partial\Omega)$, $2 < p < 2 + \frac{4}{\max(0,d-3)} + \varepsilon$, and in $L^p(\partial\Omega)$, $1 + \max(0, \frac{d-3}{d+1} - \varepsilon) < p < 2$.

The L^p problems

Conjecture

Let A be a self-adjoint, t-independent, elliptic matrix of coefficients. Then there are solutions to the L^p-Neumann problem, $1+\max(0,\frac{d-3}{d+1}-\varepsilon)<p<2+\varepsilon$,

$$\begin{cases} \nabla^m\cdot A\nabla^m w=0 \text{ in } \mathbb{R}^d_+, \quad \dot M_A w=\dot g \text{ on } \partial\mathbb{R}^d_+,\\ \|\mathcal{A}_2^+(t\nabla^m\partial_t w)\|_{L^p(\mathbb{R}^{d-1})}+\|\widetilde N_+(\nabla^m w)\|_{L^p(\mathbb{R}^{d-1})}\lesssim\|\dot g\|_{L^p(\mathbb{R}^{d-1})}\end{cases}$$

and the $\dot W^{-1,p}$-Neumann problem, $2-\varepsilon<p<2+\frac{4}{\max(0,d-3)}+\varepsilon$,

$$\begin{cases} \nabla^m\cdot A\nabla^m v=0 \text{ in } \mathbb{R}^d_+, \quad \dot M_A v=\dot h \text{ on } \partial\mathbb{R}^d_+,\\ \|\mathcal{A}_2^+(t\nabla^m v)\|_{L^p(\mathbb{R}^{d-1})}+\|\widetilde N_+(\nabla^{m-1} v)\|_{L^p(\mathbb{R}^{d-1})}\lesssim\|\dot h\|_{\dot W^{-1,p}(\mathbb{R}^{d-1})}\end{cases}$$

that are unique up to adding polynomials.

Open questions

We would like to solve the Dirichlet problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \nabla^{m-1} u = \dot{f} \text{ on } \partial\mathbb{R}^d_+, \\ \displaystyle\int_{\mathbb{R}^d_+} |\nabla^m \partial_t u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m u)(x)^2 \, dx \lesssim \|\dot{f}\|^2_{\dot{W}^2_1(\partial\mathbb{R}^d_+)}. \end{cases}$$

Open questions

We would like to solve the Dirichlet problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \nabla^{m-1} u = \dot{f} \text{ on } \partial\mathbb{R}^d_+, \\ \displaystyle\int_{\mathbb{R}^d_+} |\nabla^m \partial_t u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m u)(x)^2 \, dx \lesssim \|\dot{f}\|^2_{\dot{W}^2_1(\partial\mathbb{R}^d_+)}. \end{cases}$$

We would like to work with systems $(Lu)_j = \displaystyle\sum_{k=1}^{N} \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A^{jk}_{\alpha\beta} \partial^\beta u)$.

Open questions

We would like to solve the Dirichlet problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \nabla^{m-1} u = \dot{f} \text{ on } \partial\mathbb{R}^d_+, \\ \displaystyle\int_{\mathbb{R}^d_+} |\nabla^m \partial_t u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m u)(x)^2 \, dx \lesssim \|\dot{f}\|^2_{\dot{W}^2_1(\partial\mathbb{R}^d_+)}. \end{cases}$$

We would like to work with systems $(Lu)_j = \displaystyle\sum_{k=1}^{N} \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A^{jk}_{\alpha\beta} \partial^\beta u)$.

We would like to solve boundary value problems in Lipschitz domains rather than $\mathbb{R}^d_+$.

Open questions

We would like to solve the Dirichlet problem

$$\begin{cases} \nabla^m \cdot A\nabla^m u = 0 \text{ in } \mathbb{R}^d_+, \\ \nabla^{m-1} u = \dot{f} \text{ on } \partial\mathbb{R}^d_+, \\ \int_{\mathbb{R}^d_+} |\nabla^m \partial_t u(x,t)|^2 \, t \, dx \, dt + \int_{\mathbb{R}^{d-1}} \widetilde{N}_+(\nabla^m u)(x)^2 \, dx \lesssim \|\dot{f}\|^2_{\dot{W}^2_1(\partial\mathbb{R}^d_+)}. \end{cases}$$

We would like to work with systems $(Lu)_j = \sum_{k=1}^{N} \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A^{jk}_{\alpha\beta} \partial^\beta u)$.

We would like to solve boundary value problems in Lipschitz domains rather than $\mathbb{R}^d_+$.

We would like to look at boundary value problems with lower order terms $\nabla^m \cdot A\nabla^m u + \nabla^{m-1} \cdot B\nabla^m u + \nabla^m \cdot C\nabla^{m-1} u + \cdots = 0$.

Thank you!

Happy birthday, Steve!