Quantum Expanders and Geometry of Operator Spaces

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Preliminary Motivation: Matricial Hahn-Banach

Consider $E \subset A$ with $\dim(E) < \infty$

Fix $N \geq 1$

Then (Roger Smith): $\forall a = [a_{ij}] \in M_N(E)$

$$\| [a_{ij}] \|_{M_N(E)} = \sup_{u: E \rightarrow M_N, \|u\|_{cb} \leq 1} \| [u(a_{ij})] \|_{M_N(M_N)}$$

Natural question: How many $u$’s are needed?
Natural question: How many \( u \)'s are needed??

Fix \( \delta > 0 \). We seek to minimize \( \text{card}(\mathcal{T}) \) over all finite sets

\[
\mathcal{T} \subset \{ u : E \to M_N | \|u\|_{cb} \leq 1 \}
\]

such that

\[
\forall a = [a_{ij}] \in M_N(E) \quad (1 - \delta) \| [a_{ij}] \|_{M_N(E)} \leq \sup_{u \in \mathcal{T}} \|[u(a_{ij})]\|_{M_N(M_N)}
\]

Or with \( C = (1 - \delta)^{-1} \)

\[
\forall a = [a_{ij}] \in M_N(E) \quad \|[a_{ij}]\|_{M_N(E)} \leq C \sup_{u \in \mathcal{T}} \|[u(a_{ij})]\|_{M_N(M_N)}
\]

Smallest \( k \) denoted by \( k_E(N, C) \)
If $A$ is exact then: $\forall \delta > 0$, for all $E$ and all $N$ large enough, a single $u$ suffices:

$\exists u : E \to M_N$ with $\|u\|_{cb} \leq 1$ such that

$$\forall a = [a_{ij}] \in M_N(E) \quad (1 - \delta)\|a_{ij}\|_{M_N(E)} \leq \|u(a_{ij})\|_{M_N(M_N)} \leq \|a_{ij}\|_{M_N(E)}$$
Using *metric entropy* in dimension $nN^2$ (note $\dim(CB(E, M_N)) = nN^2$), we find:

**Proposition**

\[ \exists T \text{ with } \text{card}(T) \leq \exp c_\delta nN^2 \]

such that

\[ \forall a = [a_{ij}] \in M_N(E) \quad (1 - \delta)\|a_{ij}\|_{M_N(E)} \leq \sup_{u \in T} \|u(a_{ij})\|_{M_N(M_N)} \]

i.e. \( k_E(N, C) \leq \exp c_\delta nN^2 \)
**Proof:** Indeed, there is a $\delta$-net $T$ in the unit ball of $CB(E, M_N)$ with
$$\text{card}(T) \leq (C/\delta)^{nN^2} \leq \exp c_\delta nN^2.$$ 
Thus $\forall u$ with $\|u\|_{cb} \leq 1$ $\exists v \in T$ such that $\|u - v\|_{cb} \leq \delta$. So:

$$\| [u(a_{ij})] \|_{M_N(M_N)} \leq \| [v(a_{ij})] \|_{M_N(M_N)} + \delta \| [a_{ij}] \|_{M_N(E)}$$

$$\leq \sup_{v \in T} \| [v(a_{ij})] \|_{M_N(M_N)} + \delta \| [a_{ij}] \|_{M_N(E)}$$

$$\| [a_{ij}] \| = \sup_{\|u\|_{cb} \leq 1} \| [u(a_{ij})] \|_{M_N(M_N)} \leq \sup_{v \in T} \| [v(a_{ij})] \|_{M_N(M_N)} + \delta \| [a_{ij}] \|$$

and hence

$$(1 - \delta) \| [a_{ij}] \|_{M_N(E)} \leq \sup_{v \in T} \| [v(a_{ij})] \|_{M_N(M_N)}$$
Thus:

$$\exists T \text{ with } \text{card}(T) \leq \exp c_\delta nN^2$$

**OUR GOAL:** A class of examples of $E$ for which this is essentially best possible, i.e.

**Theorem**

*For these spaces $E$, for any such $T$ for all $N$ large enough*

$$\text{card}(T) \geq \exp c'_\delta nN^2$$

Such spaces are "extremely not exact"

Including

$$E = \text{span}[U_1, \cdots, U_n] \subset C^*(\mathbb{F}_n)$$

and

$$E = OH_n$$

These are Operator space analogues of $\ell_1^n$ and $\ell_2^n$
The term “Quantum Expander” was introduced by Hastings and by Ben-Aroya and Ta-Shma in 2007 to designate a sequence \( \{ U^{(N)} \mid N \geq 1 \} \) of \( n \)-tuples \( U^{(N)} = (U_1^{(N)}, \ldots, U_n^{(N)}) \) of \( N \times N \) unitary matrices such that there is an \( \varepsilon > 0 \) satisfying the following “spectral gap” condition: \( \forall N \forall x \in M_N \text{ with } \text{tr}(x) = 0 \)

\[
\| \sum_{1}^{n} U_j^{(N)} x U_j^{(N)*} \|_2 \leq n(1 - \varepsilon)\|x\|_2, \tag{1}
\]

where \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm on \( M_N \).

Suffices to have this for infinitely many \( N \)'s i.e. \( N \in \{ N(1) < N(2) < N(3) < \cdots \} \)

\( \exists \) Computer science motivation

An \( n \)-tuple \( U^{(N)} \) satisfying (1) will be called an \( \varepsilon \)-quantum expander.
The $\varepsilon$-quantum expander condition can be rewritten

$$\left\| \left( \sum_{j=1}^{n} U_j^{(N)} \otimes \overline{U_j^{(N)}} \right) (1 - P) \right\| \leq n(1 - \varepsilon)$$

where $P$ is the orthogonal projection onto $\mathcal{C}l$ ($l = \sum e_j \otimes \bar{e}_j$)
In analogy with the classical expanders, one seeks to exhibit (and hopefully to construct explicitly) sequences
\[ \{U^{(N_m)} \mid m \geq 1\} \]
of \(n\)-tuples of \(N_m \times N_m\) unitary matrices that are \(\varepsilon\)-quantum expanders
\[
\left\| \left( \sum_{j=1}^{n} U_j^{(N_m)} \otimes \overline{U_j^{(N_m)}} \right) (1 - P) \right\| \leq n(1 - \varepsilon)
\]
with \(N_m \to \infty\) while \(n\) and \(\varepsilon > 0\) remain fixed
Quantum expanders = non-commutative version of classical expanders

When $G$ is a finite group generated by $S = \{t_1, \cdots, t_n\}$ the associated Cayley graph $\mathcal{G}(G, S)$ is said to have a spectral gap if the left regular representation $\lambda_G$ satisfies

$$\left\| \sum \lambda_G(t_j) \right\|_{\mathbb{I}^\perp} < n(1 - \varepsilon)$$

(2)

where $\mathbb{I}$ denotes the constant function 1 on $G$. Obviously, this is equivalent to the condition that the unitaries $U_j = \lambda_G(t_j)$ satisfy (here $N = |G|$)

$$\left\| \sum_1^n U_j^{(N)} x U_j^{(N)*} \right\|_2 \leq n(1 - \varepsilon) \|x\|_2$$

when restricted to diagonal matrices $x$ with $\text{tr}(x) = 0$
A sequence of Cayley graphs $G(G^{(m)}, S^{(m)})$ constitutes an expander in the usual sense if

$$\left\| \sum_{j=1}^{n} \lambda_G(t_j)_{\| \cdot \|_{\bot}} \right\| < n(1 - \varepsilon)$$

is satisfied with $\varepsilon > 0$ and $n$ fixed while $|G^{(m)}| \to \infty$. Expanders (equivalently expanding graphs) have been extremely useful, especially (in the applied direction) since Margulis 1973 and Lubotzky-Phillips-Sarnak 1988 ("Ramanujan graphs") obtained explicit constructions. But there are also useful random ones, cf. Joel Friedman’s work Memoirs AMS 2008. See the recent survey: S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications Bull. Amer. Math. Soc. 43 (2006), 439-561.
Moreover:

Let $G$ be a finite group generated by $S = \{t_1, \cdots, t_n\}$.

Assume

$$\| \sum \lambda_G(t_j)|I_\perp\| < n(1 - \varepsilon)$$

Then for any non-trivial irreducible representation $\pi$ of $G$

$$U_j = \pi(t_j)$$

is an $\varepsilon$-quantum expander i.e.

$$\| \sum U_j \otimes \overline{U}_j(1 - P)\| \leq n(1 - \varepsilon)$$

Proof: $[\pi \otimes \overline{\pi}]|I_\perp \subset \lambda_G|I_\perp$
Moreover, if $\sigma \neq \pi$ is another \textbf{irreducible} representation, let

$$U_j = \pi(t_j) \quad V_j = \sigma(t_j)$$

Then

$$\| \sum U_j \otimes \overline{V}_j \| \leq n(1 - \varepsilon)$$

(“\$\varepsilon\$-separated”)

\textbf{Proof:} $[\pi \otimes \bar{\sigma}] \subset \lambda_G|_{\mathbb{I}^\perp}$

One can show (de la Harpe-Robertson-Valette 1993) that in the presence of an $n$-element $\varepsilon$-expander in $G (n, \varepsilon > 0$ fixed)

$$\forall N' \quad \text{card}\{\pi \mid d_\pi \leq N'\} \leq \exp(\ c_\varepsilon nN'^2)$$

\textbf{Open problem (Meshulam-Wigderson):}
Is this bound optimal?
More generally for any subset \( T \subset U(N)^n \)
such that

\[ \forall u \neq v \in T \quad \| \sum_j u_j \otimes \bar{v}_j \| \leq n(1 - \varepsilon) \]

("\( \varepsilon \)-separated")

we must have

\[ \text{card}(T) \leq \exp(c_\varepsilon nN^2) \]

\( c_\varepsilon \) depending only on \( \varepsilon > 0 \)

**Proof:** easy metric entropy argument: ambient dimension is

\( nN^2 \)
Main result

Theorem

\[ \exists \beta > 0 \quad \exists \delta > 0 \quad \text{such that} \]
for each \( 0 < \varepsilon < 1 \), for all sufficiently large integers \( n \) and \( N \),
(\text{precisely} \quad \forall n \geq n_0(\varepsilon) \quad \text{and} \quad N \geq N_0(\varepsilon, n))
there is a \( \delta \)-separated family \( \{ u(t) \mid t \in T \} \subset U(N)^n \) formed of \( \varepsilon \)-quantum expanders such that

\[ |T| \geq \exp \beta nN^2. \]
We make crucial use of a result due to Hastings 2007:

Lemma (Hastings)

If we equip $U(N)^n$ with its normalized Haar measure $\mathbb{P}$, then for each $n$ and $\varepsilon > 0$

$$\lim_{N \to \infty} \mathbb{P}\{u = (u_j) \in U(N)^n \mid \|\sum_{j=1}^{n} u_j \otimes \bar{u}_j\| \leq 2\sqrt{n - 1} + \varepsilon n\} = 1.$$ 

Best possible: This Lemma fails if $2\sqrt{n - 1}$ is replaced by any smaller number. However, our paper includes a quicker proof of a result that suffices for our needs (where $2\sqrt{n - 1}$ is replaced by $4C\sqrt{n}$, $C$ being a numerical constant).
Definition

Fix $\delta > 0$. We will say that $x, y$ in $M_N^\otimes$ are $\delta$-separated if

$$\left\| \sum x_j \otimes \bar{y}_j \right\| \leq (1 - \delta) \left\| \sum x_j \otimes \bar{x}_j \right\|^{1/2} \left\| \sum y_j \otimes \bar{y}_j \right\|^{1/2}.$$  

A family of elements is called $\delta$-separated if any two distinct members in it are $\delta$-separated.

Recall

$$\left\| \sum x_j \otimes \bar{y}_j \right\| = \sup \left\{ \left\| \sum x_j \xi \xi^* \right\|_2 \mid \left\| \xi \right\|_2 \leq 1 \right\}$$
\[ \forall x, y \in M^n_N \]

\[ x.y = \sum x_j \otimes y_j \quad x.\bar{y} = \sum x_j \otimes \bar{y}_j \]

With this notation, \( \delta \)-separated means

\[ \|x.\bar{y}\| \leq (1 - \delta)\|x.\bar{x}\|^{1/2}\|y.\bar{y}\|^{1/2} \]

Note that if \( x \in U(N)^n \) i.e. \( x_j \) is unitary then

\[ \|x.\bar{x}\| = n \]

because \( \sum_{j=1}^n x_j^* x_j = nI \)
Operator spaces

**Definition**

(“Non-commutative Banach spaces”) An operator space is a subspace of $B(\mathcal{H})$, i.e. we are given

$$E \subset B(H)$$

Morphisms: CB maps

**Fundamental result:** Arveson version of Hahn-Banach CB maps, *Acta* 1969

“Operator space Theory" is now well developed after Ruan’s 1987 thesis cf. Effros-Ruan, Blecher-Paulsen, and many more...

cf. one book by Effros-Ruan, & one by myself
Let
\[ u : E \rightarrow F \]
be a linear map between operator spaces. We denote for any given \( N \geq 1 \)
\[ u_N = Id \otimes u : M_N(E) \rightarrow M_N(F) \]
\[ [a_{ij}] \mapsto [u(a_{ij})] \]
\[ (u_N = Id \otimes u : M_N \otimes E \rightarrow M_N \otimes F \]
Recall that
\[ \|u\|_{cb} = \sup_{N \geq 1} \|u_N\| \].
For an operator space the norm is replaced by a sequence of norms

$$\\{\|\cdot\|_{M_N(E)} \mid N \geq 1\}$$

The ordinary norm on $E$ corresponds to $N = 1$.
Minimal Tensor Product

Given

\[ E \subset B(H) \quad F \subset B(\mathcal{H}) \]

we define

\[ E \otimes_{\text{min}} F \subset B(H \otimes_2 \mathcal{H}) \]

(“spatial” or “minimal” tensor product)

Note: This norm will be used everywhere!
A first application related to non-compactness/non-separability of the metric space of $n$-dim operator spaces (Junge-P, 1994)

**Theorem**

There are numbers $\beta_1 > 0$, $\delta_3 > 0$, $n_0 > 1$ and a function $n \mapsto N_0(n)$ from $\mathbb{N}$ to itself such that for any $n \geq n_0$ and $N \geq N_0(n)$, there is a family $\{E_t \mid t \in T_1\}$ of $n$-dimensional subspaces of $M_N$, with cardinality $|T_1| \geq \exp \beta_1 nN^2$, such that for any $s \neq t \in T_1$ we have

$$d_{cb}(E_s, E_t) > 1 + \delta_3.$$

$$d_{cb}(E, F) = \inf \{\|u\|_{cb}\|u^{-1}\|_{cb} \mid u : E \to F\}$$
An application with Geometric flavor

Operator space analogue of Hilbert space (or Euclidean space)

\[ \text{OH} = \text{span}[\theta_j \mid j \in \mathbb{N}] \subset B(H) \]

is analogous to \( \ell_2 \)

\[ \text{OH characterized by:} \]

For any \( N \) and any (finite) sequence \( x = (x_j) \) with \( x_j \in M_N \)

\[ \| \sum x_j \otimes \theta_j \| = \| \bar{x} \cdot x \|^1/2 = \sup \| \bar{y} \cdot x \|. \]

Similarly:

\[ \text{OH}_n = \text{span}[\theta_j \mid 1 \leq j \leq n] \subset B(H) \]

is analogous to \( \ell_n^2 \)
Definition of $OH$ uses Haagerup’s Cauchy-Schwarz inequality

$$\forall x = (x_j) \in B(H)^{(\mathbb{N})} \quad \|\bar{y}.x\| \leq \|\bar{y}.y\|^{1/2} \|\bar{x}.x\|^{1/2}$$

and hence $\exists \theta_j$ such that

$$\|\sum x_j \otimes \theta_j\| = \|\bar{x}.x\|^{1/2} = \sup_{y:\|\bar{y}.y\|\leq 1} \|\bar{y}.x\|.$$ 

In particular, if $\dim(H) = N$ we have

$$\forall x = (x_j) \in M_n^N \quad \|\sum_{1}^{n} x_j \otimes \theta_j\| = \|\bar{x}.x\|^{1/2} = \sup_{y:\|\bar{y}.y\|\leq 1} \|\bar{y}.x\|.$$ 

When $N = 1$, we recover the classical formula

$$\|\sum x_j \otimes e_j\| = \langle x, x \rangle^{1/2} = \sup_{y:\langle y, y \rangle\leq 1} |\langle y, x \rangle|.$$
Quantum Expanders and Geometry of Operator Spaces

Classical Geometric problem
Consider a symmetric convex body $B = B_E$
Unit ball of an $n$-dimensional normed space $E$
Given a constant $C > 1$, estimate the minimal number $k = k_E(C)$ of functionals $f_1, \ldots, f_k$ in the dual $E^*$ such that

$$\forall x \in E \quad \sup_{1 \leq j \leq k} |f_j(x)| \leq \|x\|_E \leq C \sup_{1 \leq j \leq k} |f_j(x)|.$$
Geometrically (in the real case): $B_E^* \simeq \text{conv}\{\pm f_j\} \\
P = \text{conv}\{\pm f_j\} = \text{polyhedron with at most } 2k \text{ vertices} \\
So the polar, $P^*$, has at most $2k$ faces. 

**Example:** the $n$-dimensional cube has $2^n$ vertices and $2n$ faces 

When $E$ has (real) dimension $n$ it is well known that 

$$\forall C = 1 + \delta \in [1, 2] \quad k_E(C) \leq \exp((\frac{4}{\delta})n).$$
Extreme cases:

- Maximal order of growth:
  \[ E = \ell_2^n \text{ (or } \ell_p^n \text{ for } 1 \leq p < \infty \text{ or uniformly convex)} \]

  \[ k_E(1 + \delta) \geq \exp(c_\delta n). \]

- Minimal order of growth:
  \[ k_E(C) = n \text{ for } E = \ell_\infty^n \]
Let $E$ finite dim. Banach space. Fix a constant $C > 1$, Recall

$$k_E(C)$$

is the minimal number $k$ of functionals $f_1, \cdots f_k \in B_{E^*}$ such that

$$\|x\|_E \leq C \sup_{1 \leq j \leq k} |f_j(x)|.$$

Equivalently:

$$k_E(C) = \inf\{k \mid E \supset \ell_k^C \}$$
Matricial version of $k_E(C)$

Let $E$ be a finite dimensional operator space. Fix $C > 0$. We denote by

$$k_E(N, C)$$

the smallest $k$ such that there are linear maps

$$f_j : E \to M_N \quad (1 \leq j \leq k)$$

with $\|f_j\|_{cb} \leq 1$ satisfying $\forall x \in M_N(E)$

$$\|x\|_{M_N(E)} \leq C \sup_{1 \leq j \leq k} \|(Id \otimes f_j)(x)\|_{M_N(M_N)}.$$ 

Equivalently:

$$k_E(N, C) = \inf\{k \mid E \subset \ell_\infty^k \otimes M_N\}$$
Theorem

There are numbers $C_1 > 1$, $b > 0$ such that for any $n, N$ large enough we have

$$k_{OH_n}(N, C_1) \geq \exp bnN^2.$$ 

Compare: Universal UPPER bound:

for any $n$-dimensional $E$

for all $N$ large enough

$$k_E(N, C_1) \leq \exp b'nN^2$$

(because $\dim(M_N(E)) = nN^2$)
We will recall a classical argument for the Euclidean case, i.e. the case $N = 1$. Let $E = \ell^n_2$. Let $C = (1 - \delta)^{-1}$ for some $0 < \delta < 1$. We will show

$$k_E(C) \geq \exp cn$$

**Two ingredients:**

(i) **Uniform smoothness** of $E^*$: For any given point $t \in B_E$, and any $f \in B_E$

$$\langle f, t \rangle \geq 1 - \delta \implies \| f - t \| \leq \sqrt{2\delta}$$

(ii) **Large metric entropy** (valid for any $E$): There is a subset $T \subset S_E$ with $|T| \geq \exp c\varepsilon' n$ such that

$$\forall s \neq t \in T \times T \quad \| s - t \| \geq \varepsilon'$$

$$\iff \Re(\bar{s} . t) \leq 1 - \varepsilon'^2 /2$$
Proof that \( k = k_E(C) \geq \exp cn \): Recall \( C = (1 - \delta)^{-1} \).

\[
\forall x \in E \sup_{1 \leq j \leq k} |f_j(x)| \leq \|x\|_E \leq (1 - \delta)^{-1} \sup_{1 \leq j \leq k} |f_j(x)|.
\]

Real case for simplicity: changing \( k \) to \( 2k \) (consider \( \pm f_j \)):

\[
\forall x \in E \sup_{1 \leq j \leq 2k} f_j(x) \leq \|x\|_E \leq (1 - \delta)^{-1} \sup_{1 \leq j \leq 2k} f_j(x).
\]

Take \( x = t \). For any \( t \in T \), there is \( j(t) \) so that

\[
1 - \delta \leq f_j(t)(t)
\]

But by the smoothness

\[
(*) \quad \|f_j(t) - t\| \leq \sqrt{2\delta}
\]

Now choosing \( \delta < \varepsilon'^2 / 32 \) and recalling \( \|s - t\| \geq \varepsilon' \) for \( s \neq t \)

\[
(*) \Rightarrow \|f_j(t) - f_j(s)\| \geq \varepsilon' - 2\sqrt{2\delta} > \varepsilon' / 2 > 0
\]

and hence \( j(s) \neq j(t) \) whenever \( s \neq t \)

**Conclusion:** \( 2k \geq |T| \) and hence \( 2k \geq \exp c \varepsilon' n \). Q.E.D.
We follow this proof to show

**Theorem**

*There are numbers $C_1 > 1$, $b > 0$ such that for any $n, N$ large enough we have*

$$k_{OH_n}(N, C_1) \geq \exp bnN^2.$$ 

Proof requires **matricial** analogues of (i) and (ii)
Recall \( \| \bar{y}.x \| = \| \sum \bar{y}_j \otimes x_j \| \)

We will need the analogue of smooth points on the sphere

Consider \( x = (x_j) \in M_N^n \) in the “unit sphere" i.e. such that \( \| \bar{x}.x \| = 1 \).

Let \( \text{Orb}(y) = \{ y' = (y'_j) \mid \exists u, v \in U(N) \; y'_j = uy_jv \} \)

Recall \( 1 = \sup_{\| \bar{y}.y \| \leq 1} \| \bar{y}.x \| \).

Note \( \| \bar{y}.x \| = 1 \implies \| \bar{y}'.x \| = 1 \) \( \forall y' \in \text{Orb}(y) \)

We say that \( x = (x_j) \in M_N^n \) with \( \| \bar{x}.x \| = 1 \) is \( M_N \)-smooth if

\[
\| \bar{y}.y \| \leq 1 \quad \| \bar{y}.x \| = 1 \implies y \in \text{Orb}(x)
\]

**Lemma (Matricial uniform smoothness)**

Consider \( x = (x_j) = n^{-1/2}(u_j) \) with \( (u_j) \in U(N)^n \). Observe \( \| \bar{x}.x \| = 1 \). Then, \( x \) is \( M_N \)-smooth IFF \( (u_j) \) is an \( \varepsilon \)-quantum expander for some \( \varepsilon > 0 \). Moreover, a lower bound on \( \varepsilon \) ensures “uniform smoothness" as in (ii).
Quantum expanders

Thus we need to work with points in the sphere of $M_N(OH_n)$ of the form: $t = \sum_j t_j \otimes \theta_j$ with $\|\bar{t}.t\| = 1$ that in addition form $\varepsilon$-quantum expanders. We identify such a point with $s = (s_j) \in M^n_N$. We will use $t_u = (t_j) = n^{-1/2}(u_j)$ with $(u_j) \in U(N)^n$. $t_u$ is just $u$ after renormalization.

Lemma (Matricial metric entropy)

There are numbers $\varepsilon' > 0$, $b > 0$ such that for any $0 < \varepsilon < 1$ for any $n, N$ large enough there is a subset

$$T \subset U(N)^n \quad \text{with} \quad |T| \geq \exp bnN^2$$

formed of $\varepsilon$-quantum expanders that are $\varepsilon'$-separated i.e. $\forall u \neq v \in T \quad n^{-1}\|\bar{u}.v\| = \|\bar{t}_u.t_v\| \leq 1 - \varepsilon'^2/2$ and a fortiori

$$(n^{-1}\sum_{j=1}^{n}\|u_j - v_j\|_{L_2(\tau_N)}^2)^{1/2} \geq \varepsilon'$$
With these lemmas, we can run the proof of

\[ k_{OH_n}(N, C_1) \geq \exp bnN^2 \]

exactly as we did for the classical case \( N = 1 \)
Similarly, in addition to $E = OH_n$, we have

$$\exp b' nN^2 \geq k_E(N, C_1) \geq \exp bnN^2$$

for

$$E = \ell_1^n \subset C^*(\mathbb{F}_n)$$

and for

$$E = R_n + C_n$$
This leads to a natural class: the operator spaces (or $C^*$-algebras) $X$ such that for some $C$ for any $E \subset X$ we have

$$\limsup_{N \to \infty} \frac{\log k_E(N, C)}{N^2} = 0$$

We call these (matricially) subGaussian. They should verify a sort of "generalized exactness". Unfortunately we could obtain significant results only with a stronger property

$$\limsup_{N \to \infty} \frac{\log k_E(N, C)}{N} = 0$$

called (matricially) subexponential.
Indeed in “(matricially) subexponential" spaces
Behaviour /Gaussian random matrices similar to exact case
⇒ Subexponential Operator spaces satisfy the OSGT
Extends the Grothendieck’s Theorem from Shlyakhtenko-P
2002 from exact to subexponential
A variant: $K_E(C, N)$

$K_E(N, C)$ the smallest $K$ such that there is a single map

$$u : E \rightarrow M_K$$

with $$\|u\|_{cb} \leq 1$$

satisfying

$$\forall a = [a_{ij}] \in M_N(E) \quad \|[a_{ij}]\|_{M_N(E)} \leq C\|[u(a_{ij})]\|_{M_N(M_K)}$$

Equivalently (instead of $E \subset \ell^k_\infty \otimes M_N$) we have

$$E \subset M_K$$

Note obviously

$$K_E(N, C) \leq Nk_E(N, C)$$

So $k_E(N, C)$ subexponential implies $K_E(N, C)$ subexponential
A exact $\iff \exists C \forall E \subset A \sup_N K_E(N, C) < \infty$

But we have examples of $C^*$-algebras $A$ such that $K_E(N, 1 + \varepsilon)$ has subexponential growth (even polynomial growth) for any $\varepsilon > 0$ and any finite dim. $E \subset A$ but $A$ is not exact.
Let
\[ U_j(\omega) = \bigoplus_N U_j^{(N)} \subset \bigoplus_N M_N \]

Let
\[ A_\omega = C^* < U_j(\omega), 1 \leq j \leq n > \quad 3 \leq n \leq \infty \]

**Theorem**

*For almost all* \( \omega \), \( A_\omega \) *is subexponential*

*i.e. \( \forall C > 1 \ \forall E \subset A \ \lim sup N^{-1} \log K_E(N, C) = 0 \)*

*but not exact.*

Initially: Gaussian case (using Haagerup-Thorbjoernsen)

Later: Unitary case: de la Salle (using Collins-Male)
Thank you!