NEW TRENDS IN HARMONIC ANALYSIS AT ICMAT

Research Term on Real Harmonic Analysis and Applications to PDE
Madrid, April - June 2013

Aula Azul (ICMat, Campus de Cantoblanco)

Tuesday 4 June 12:00-13:00 Henri Martikainen
Tuesday 4 June 15:00-16:00 Mihalis Mourgoglou
Wednesday 5 June 10:00-12:00 Tatiana Toro
Wednesday 5 June 12:00-13:00 David Rule
Wednesday 5 June 15:00-17:00 Svitlana Mayboroda
Thursday 6 June 15:00-16:00 Irina Mitrea (I.)
Thursday 6 June 16:00-17:00 Irina Mitrea (II.)
Friday 7 June 12:00-13:00 Dmitriy Bilyk

Abstracts:

RECENT BOUNDEDNESS RESULTS FOR SQUARE FUNCTIONS AND CALDERÓN–ZYGMUND OPERATORS: NON-HOMOGENEOUS AND BI-PARAMETER THEORY
Henri Martikainen, Université Paris-Sud 11

We discuss a selection of recent boundedness results (\(T^1\) or \(T^b\) theorems) for square functions and Calderón–Zygmund operators. The focus is on operators defined in the product space \(\mathbb{R}^n \times \mathbb{R}^m\) (bi-parameter theory) or on operators defined using general measures \(\mu\) (non-homogeneous theory) – or both! Time permitting we discuss a new simple proof technique based on averaging identities over good Whitney regions.
RELLICH ESTIMATES AND EXTRAPOLATION OF SOLVABILITY OF ELLIPTIC SYSTEMS
Mihalis Mourgoglou, Université Paris-Sud

We study extrapolation of regularity and Neumann problems for elliptic systems in divergence form in the upper half-space assuming De Giorgi type conditions. We shall base our study of solvability on energy solutions. For those solutions, using recent and proving new results for boundary layer potentials, we shall show equivalence of the $L^p$ norm of (a modified) non-tangential maximal function of $\nabla u$ with $L^p$ (if $p > 1$) or $H^p$ norms (if $p \leq 1$) in a range of $p$ of both the tangential gradient and the conormal derivative at the boundary. Solvability amounts to controlling one of the boundary norms by the other (Rellich inequality). Our extrapolation method then amounts to extrapolating these boundary inequalities using atomic Hardy spaces and interpolation. This is a joint work with P. Auscher and S. Hofmann.

REGULARITY FOR ALMOST MINIMIZERS WITH FREE BOUNDARY
Tatiana Toro, University of Washington

In recent work with Guy David we introduce the notion of almost minimizer for a series of functionals previously studied by Alt-Caffarelli and Alt-Caffarelli-Friedman. We prove regularity results for these almost minimizers and explore the structure of the corresponding free boundary. A key ingredient in the study of the 2-phase problem is the existence of almost monotone quantities. The goal of this talk is to present these results in a self-contained manner, emphasizing both the similarities and differences between minimizers and almost minimizers.

BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC EQUATIONS SATISFYING A CARLESON MEASURE CONDITION
David Rule, Linköpings universitet

We study solutions $u: \Omega \to \mathbb{R}$ to the divergence form elliptic equation $\sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) = 0$ in a Lipschitz domain $\Omega \subset \mathbb{R}^n$. We show that the Neuman and regularity problems are solvable with data in $L^p(\partial\Omega)$ (for a fixed $p \in (1, \infty)$) provided that

$$\sum_{i,j,k=1}^{n} |\partial_k a_{ij}(x)|^2 \text{dist}(x, \partial\Omega) dx$$

is a Carleson measure with sufficiently small norm. The same conclusion holds when the derivatives in [1] are replaced by an appropriate averaging condition. This compliments the results of Dindoš-Petermichl-Pipher regarding the Dirichlet problem. This is joint work with Martin Dindoš and Jill Pipher.
WELL-POSEDNESS IN $L^p$ FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

Svitlana Mayboroda, University of Minnesota

One of the simplest and the most important results in elliptic theory is the maximum principle. It provides sharp estimates for the solutions to elliptic PDEs in $L^\infty$ in terms of the corresponding norm of the boundary data. It holds on arbitrary domains for all (real) second order divergence form elliptic operators $-\text{div} A \nabla$. The well-posedness of boundary problems in $L^p$, $p < \infty$, is a far more intricate and challenging question, even in a half-space, $\mathbb{R}^{n+1}_+$. In particular, it is known that some smoothness of $A$ in $t$, the transversal direction to the boundary, is needed.

We will discuss the state of the art in elliptic theory surrounding such well-posedness questions and then concentrate on the recent results for elliptic PDEs associated to matrices $A$ independent on the transversal direction to the boundary. We demonstrate that for any real (not necessarily symmetric) $t$-independent matrix $A$ there exists $p < \infty$ such that the Dirichlet boundary value problem is well-posed in $L^p$, and that the corresponding Regularity problem is well-posed in the dual range. The well-posedness extends to the perturbations of real $t$-independent matrices, in $L^\infty$, or in the sense of Carleson measures. Such a result was only known in the setting of real symmetric matrices (D. Jerison, C. Kenig, 1981). The non-symmetric case was open since then, and ultimately had to be approached by completely different techniques, relying, in particular, on the Kato problem. In 2000 Kenig, Koch, Pipher, and Toro established the well-posedness of Dirichlet problem for non-symmetric matrices in dimension 2. The present work pertains to all dimensions $n \geq 2$.

This is joint work with S. Hofmann, C. Kenig, and J. Pipher.

(I.) ON THE EXTENSION PROBLEM IN NONTANGENTIALLY ACCESSIBLE DOMAINS WITH AHLFORS-DAVID REGULAR BOUNDARIES

Irina Mitrea, Temple University

Historically, the development of modern Harmonic Analysis has been inexorably linked with the theory of Complex Variables in the plane. While subsequent real variables methods have permitted generalizations to higher dimensions, the interplay between Harmonic Analysis and Complex Analysis remains strong even in the higher dimensional setting. Of course, this presupposes working with functions of several complex variables or with other notions of analyticity, amenable to the higher dimensional case. In this two talks I will exemplify the intricate nature of such connections by presenting progress on two problems originating in Complex Analysis which due to developments in Harmonic Analysis can be now solved in significantly more general settings than originally anticipated.

Hartogs’ discovery in 1906 of the astonishing phenomenon that holomorphic functions fill compact holes in $\mathbb{C}^n$ with $n \geq 2$ marked the birth of multidimensional complex analysis and over the years, extending analytic objects has been a recurring theme in Complex Variables. In this talk I will discuss recent progress, joint work with Marius Mitrea, in establishing a version of Hartogs’ theorem involving functions that satisfy tangential Cauchy-Riemann equations on the boundary of NTA domains with Ahlfors-David regular boundaries. A key ingredient in our approach is the study of the Bochner-Martinelli singular integral operator in rough domains.
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The second lecture focuses on Recent Progress in the Riemann-Hilbert Problem for Dirac Operators in Uniformly Rectifiable Domains. In its classical form, the Riemann-Hilbert problem asks for determining two holomorphic functions defined on either side of a surface $\Sigma$, satisfying a boundary condition of transmission type along $\Sigma$ involving a symbol function $\Phi$. In this regard, I will report on recent progress with Marius Mitrea and Michael Taylor describing the Fredholm solvability in the most geometric measure theoretic setting in which such a problem is meaningfully formulated. This involves replacing a complex plane by a Riemannian manifold $\mathcal{M}$, the surface $\Sigma$ by a uniformly rectifiable subset of $\mathcal{M}$, and the Cauchy-Riemann operator by a general Dirac operator on $\mathcal{M}$ with low regularity assumptions on its coefficients. This topic interfaces with Index Theory on manifolds, and as an application I will discuss the most general Bojarski index formula known to date.

HARMONIC ANALYSIS METHODS IN DISCREPANCY THEORY
Dmitriy Bilyk, University of Minnesota

Geometric discrepancy theory is concerned with different variations of the following question: how well can one approximate a continuous distribution by a discrete one and what are the limitations that necessarily arise in such approximations. Historically, the methods of harmonic analysis (Fourier transform, Fourier series, wavelets, Riesz products, etc.) have played a pivotal role in the subject. I will give an overview of the problems, methods, and results in the field and discuss some recent developments, as well as connections with other areas of mathematics (approximation theory, probability etc).