Iterative Regularization Methods for Inverse Problems: Lecture 3

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Outline

framework

operator equation:

$$Y = F(f^\dagger) + \sigma\xi + \delta\zeta$$

$F : D(F) \subset \mathcal{X} \to \mathcal{Y}$ Fréchet differentiable and one-to-one, $\mathcal{X}, \mathcal{Y}$ separable Hilbert spaces.

ξ	normalized stochastic noise (a Hilbert space process in $\mathcal{Y}$)
$\sigma \geq 0$	stochastic noise level
$\zeta \in \mathcal{Y}$	normalized deterministic noise, $\|\zeta\| = 1$
$\delta \geq 0$	deterministic noise level

monographs for the deterministic case $\sigma = 0$:

A.B. Bakushinskiĭ, M.Y. Kokurin. *Iterative Methods for Approximate Solution of Inverse Problems*. Springer, Dordrecht, 2004.

B. Kaltenbacher, A. Neubauer, O. Scherzer. *Iterative Regularization Methods for Nonlinear Ill-Posed Problems.* Radon Series on Computational and Applied Mathematics, de Gruyter, Berlin, 2008

nonlinear Tikhonov regularization

- Tikhonov regularization can immediately be generalized to nonlinear inverse problems:

$$\widehat{f}_\alpha = \mathrm{argmin}_{f\in D(F)} \left(\|F(f) - Y\|^2 + \alpha \|f - f_0\|^2 \right)$$

- Sufficient condition for existence of a global minimum: F weakly sequentially closed, i.e. if $\{f_n : n \in \mathbb{N}\} \subset D(F)$, $f_n \overset{n\to\infty}{\rightharpoonup} f \in \mathcal{X}$ and $F(f_n) \overset{n\to\infty}{\rightharpoonup} g \in \mathcal{Y}$, then $f \in D(F)$ and $F(f) = g$.
- difficulties:
  - A global minimum is not necessarily unique.
  - There may be many local minima.

regularizing property of nonlinear Tikhonov regularization

Theorem

Assume further that F is weakly sequentially closed and injective, and let $f^\dagger \in D(F)$. Consider a sequence $g^{\delta_n} \in \mathcal{Y}$ such that $\|g^{\delta_n} - F(f^\dagger)\| \leq \delta_n$ and $\delta_n \to 0$ and let $f_{\alpha_n}^{\delta_n}$ be global minimizers of

$$J_n(f) := \|F(f) - g^{\delta_n}\|_{\mathcal{Y}}^2 + \alpha_n \|f - f_0\|_{\mathcal{X}}^2, \qquad f \in D(F).$$

Moreover, assume that the regularization parameters $\alpha_n > 0$ are chosen such that

$$\alpha_n \to 0, \qquad \frac{\delta_n^2}{\alpha_n} \to 0 \qquad \text{as } n \to \infty.$$

Then

$$\|f_{\alpha_n}^{\delta_n} - f^\dagger\|_{\mathcal{X}} \to 0, \qquad n \to \infty\,.$$

source conditions for nonlinear inverse problems

- Recall that in general convergence can be arbitrarily slow for ill-posed problems without further a-priori information.
- For nonlinear inverse problems smoothness conditions are usually formulated as nonlinear source conditions of the form

$$f^\dagger - f_0 = \varphi(F'[f^\dagger]^* F'[f^\dagger])w \qquad \text{for some } w \in \mathcal{X}$$

  with an index function φ (i.e. φ continuous, monotonically increasing, $\varphi(0) = 0$)
- For $\varphi(t) = t^{1/2}$ this is equivalent to

$$f^\dagger - f_0 = F'[f^\dagger]^* w \qquad \text{for some } w \in \mathcal{Y}.$$

- Often $\|w\|$ is assumed to be sufficiently small.

convergence rate results for nonlinear Tikhonov regularization

deterministic errors:

- H.W. Engl, K. Kunisch, A. Neubauer. *Convergence rates for Tikhonov regularization of nonlinear ill-posed problems*. **Inverse Problems** 5:523–540, 1989.
- A. Neubauer. *Tikhonov regularization of nonlinear ill-posed problems in Hilbert scales.* **Appl. Anal.** 46:59–72, 1992.
- O. Scherzer, H.W. Engl, K. Kunisch. *Optimal a-posteriori parameter choice for Tikhonov regularization for solving nonlinear ill-posed problems*. **SIAM J. Numer. Anal.** 30:1796–1838, 1993.
- U. Tautenhahn, Q. Jin. *Tikhonov regularization and a posteriori rules for solving nonlinear ill posed problems.* **Inverse Problems** 19:1–21, 2003.

random noise:

- F. O'Sullivan, *Convergence characteristics of method of regularization estimators for nonlinear operator equations.* **SIAM J. Num. Anal** 27:1635–1649, 1990.
- N. Bissantz, T. Hohage, A. Munk. *Consistency and rates of convergence of nonlinear Tikhonov regularization with random noise.* **Inverse Problems** 20:1773–1791, 2004.
- T. Hohage, M. Pricop. *Nonlinear Tikhonov regularization in Hilbert scales for inverse boundary value problems with random noise.* **Inv. Probl. Imag.** 2:271–290, 2008.

iterative regularization methods

An iterative method $\widehat{f}_{k+1} := \Phi(\widehat{f}_k, \dots, \widehat{f}_0, Y)$ together with a stopping rule $N(\delta, Y)$ is called an iterative regularization method for F if for all $f^\dagger \in D(F)$ all Y satisfying $\|Y - F(f^\dagger)\| \le \delta$ and all initial guesses f_0 sufficiently close to $f^\dagger$ the following conditions hold:

- For exact data ($\delta = 0$) either $N = N(\delta, Y) < \infty$ and $\widehat{f}_K = f^\dagger$ or $K = \infty$ and $\|f_k - f^\dagger\| \to 0$ for $n \to \infty$.
- For data perturbed by deterministic noise we require that

$$\sup_{\|Y - F(f^\dagger)\| \le \delta} \|\widehat{f}_{N(\delta,Y)} - f^\dagger\| \to 0, \qquad \delta \to 0.$$

- For data perturbed by random noise we require that

$$\mathbb{E}\|\widehat{f}_{N(Y)} - f^\dagger\|^2 \to 0, \qquad \sigma \to 0.$$

nonlinear Landweber iteration

- Idea: Minimize the data misfit functional

$$\Phi(f) := \frac{1}{2}\|F(f) - Y\|^2.$$

- Since

$$\Phi'[f]h = \langle F(f) - Y, F'[f]h\rangle = \langle F'[f]^*(F(f) - Y), h\rangle ,$$

  the direction of steepest descent is $-F'[f]^*(F(f) - Y)$.
- This leads to the iteration formula

$$f_{k+1}^{\delta} = f_k^{\delta} - \mu F'[f_k^{\delta}]^*(F(f_k^{\delta}) - Y)$$

  with a step size parameter $\mu > 0$.
- For Landweber iteration, μ is fixed with $\mu\|F'[f]\|^2 \leq 1$ for all f in a neighborhood of $f^{\dagger}$.

convergence of Landweber iteration

Theorem

If the tangential cone condition

$$\|F(f) - F(\bar{f}) - F'[f](f - \bar{f})\|_{\mathcal{Y}} \leq \eta \|F(f) - F(\bar{f})\|_{\mathcal{Y}}$$

holds for all $f, \bar{f}$ in a neighborhood of $f^{\dagger}$ and some $\eta < \frac{1}{2}$, then Landweber iteration together with the discrepancy principle with $\tau > 2\frac{1+\eta}{1-2\eta}$ is a regularization method in the sense of the previous definition.

M. Hanke, A. Neubauer, O. Scherzer. *A convergence analysis of the Landweber iteration for nonlinear ill-posed problems.* **Numer. Math.** 72:21–37, 1995.

nonlinearity vs. degree of ill-posedness

- If F' is Lipschitz continuous with constant L, the Taylor remainder satisfies the estimate

$$\left\| F(f+h) - F(f) - F'[f]h \right\|_{\mathcal{Y}} \leq \frac{L}{2} \|h\|_{\mathcal{X}}^2, \qquad \|h\| \to 0$$

- Suppose $h = r_\alpha((F'[f]^* F'[f]))(F'[f]^* F'[f])^\nu w$. Then
  - $\|F'[f]h\|_{\mathcal{Y}} = \mathrm{O}\left(\alpha^{\nu+\frac{1}{2}}\right)$
  - $\|h\|_{\mathcal{X}}^2 = \mathrm{O}\left(\alpha^{2\nu}\right)$
- This estimate of the Taylor remainder is only useful for $\nu \geq \frac{1}{2}$!
- It is an important open question if the tangential cone condition (or other conditions on the degree of nonlinearity of F) hold e.g. for the conductivity problem or inverse scattering problems.

Newton's method

standard Newton method: Compute an update $h_k = \widehat{f}_{k+1} - \widehat{f}_k$ by solving the linearized equation

$$F'[f_k^\delta]h_k = Y - F(f_k^\delta).$$

Since the linearized equation typically inherits the ill-posedness of the nonlinear equation, it must be regularized!

Levenberg-Marquardt and IRGNM

- Tikhonov applied to kth Newton equation leads to minimization of the quadratic functional

$$h \mapsto \|F'[f_k^\delta]h + F(f_k^\delta) - Y\|_{\mathcal{Y}}^2 + \alpha_k \|h\|_{\mathcal{X}}^2 .$$

  - Formula coincides with Levenberg-Marquardt algorithm, one of the most efficient methods for least-squares problems in optimization
  - In optimization the parameters α_k are determined by a trust-region philosophy.
  - Often for inverse problems the simple choice $\alpha_k = \alpha_0 q^k$ for some $q \in (0,1)$ (e.g. $q = 1/2$ or $q = 2/3$) works best.
- Iteratively Regularized Gauss-Newton Method (IRGNM): Update $h_k \in \mathcal{X}$ is the unique minimum of the quadratic functional

$$h \mapsto \|F'[f_k^\delta]h + F(f_k^\delta) - Y\|_{\mathcal{Y}}^2 + \alpha_k \|h + f_k^\delta - f_0\|_{\mathcal{X}}^2$$

A. B. Bakushinskiĭ. *The problem of the convergence of the iteratively regularized Gauss-Newton method.* **Comput. Maths. Math. Phys.** 32:1353–1359, 1992.

IRGNM for linear problems

- If $F = T$ is linear, f_{k+1}^{δ} is minimizer of

$$f \mapsto \|Tf - Y\|_{\mathcal{Y}}^2 + \alpha_k \|f - f_0\|_{\mathcal{X}}^2 = \text{min!}$$

- Bias and variance must be balanced by proper choice of the stopping index.
- Optimal rates can only be expected for Hölder smoothness classes with index $\nu \leq 1$!

regularizing property of Levenberg-Marquardt algorithm

Theorem

The Levenberg-Marquardt algorithm together with the discrepancy principle is an iterative regularization method if there exists a constant $c > 0$ such that

$$\|F(f) - F(\bar{f}) - F'[f](f - \bar{f})\|_{\mathcal{Y}} \leq c\|f - \bar{f}\|_{\mathcal{X}} \, \|F(f) - F(\bar{f})\|_{\mathcal{Y}}$$

for all $f, \bar{f}$ in a neighborhood of the solution.

M. Hanke. *A regularizing Levenberg-Marquardt scheme, with applications to inverse groundwater filtration problems.*, **Inverse Problems** 13:79–95, 1997.

regularizing property of IRGNM

Theorem

The IRGNM with the discrepancy principle is a regularization method if there exist linear operators $R(\bar{f}, f), Q(\bar{f}, f)$ and constants $\gamma_R, \gamma_Q > 0$ such that

$$
\begin{aligned}
F'[\bar{f}] &= R(\bar{f}, f)F'[f] + Q(\bar{f}, f) \\
\|I - R(\bar{f}, f)\| &\leq \gamma_R \|\bar{f} - f\|, \\
\|Q(\bar{f}, f)\| &\leq C_Q \|F'[f^\dagger](\bar{f} - f)\|
\end{aligned}
$$

for all all $f, \bar{f}$ in a ball around $f^\dagger$.

B. Blaschke/Kaltenbacher, A. Neubauer, O. Scherzer. *On convergence rates for the iteratively regularized Gauss-Newton Method.* **IMA J. Num. Anal.** 17:421–436, 1997.

rates of convergence for IRGNM

For Hölder source conditions with $\nu \geq 1/2$ no nonlinearity condition (except Lipschitz continuity of F') is required!

A. B. Bakushinskiĭ. *The problem of the convergence of the iteratively regularized Gauss-Newton method.* **Comput. Maths. Math. Phys.** 32:1353–1359, 1992.
Hölder source conditions with $\nu = 1$

B. Blaschke/Kaltenbacher, A. Neubauer, O. Scherzer. *On convergence rates for the iteratively regularized Gauss-Newton Method.* **IMA J. Num. Anal.** 17:421–436, 1997.
Hölder source conditions with $0 \leq \nu < 1$

T. Hohage, *Logarithmic convergence rates of the iteratively regularized Gauss-Newton method for an inverse potential and an inverse scattering problem.* **Inverse Problems** 13:1279–1299, 1997.
logarithmic source conditions

Newton-type methods with other linear regularization methods

IRGNM with Tikhonov replaced by some method described by filter functions $\{q_\alpha\}$ with possibly higher qualification:

$$f_{k+1}^\delta := f_0 + q_{\alpha_k}\left(T_k^* T_k\right) T_k^* \Big(Y - F(f_k^\delta) + T_k(f_k^\delta - f_0) \Big), \qquad T_k := F'[f_k^\delta]$$

- A. B. Bakushinskiĭ. *The problem of the convergence of the iteratively regularized Gauss-Newton method.* **Comput. Maths. Math. Phys.** 32:1353–1359, 1992.
- B. Kaltenbacher. *Some Newton-type methods for the regularization of nonlinear ill-posed problems*, **Inverse Problems** 13: 729–753, 1997.
- B. Kaltenbacher. *A posteriori parameter choice strategies for some Newton type methods for the regularization of nonlinear ill-posed problems*, **Numer. Math** 79:501–528, 1998.

Levenberg-Marquardt-type methods with other linear methods:

- M. Hanke. *Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems.* **Numer. Funct. Anal. Optim.** 18:971–993, 1997.
- A. Rieder. *On the regularization of nonlinear ill-posed problems via inexact Newton iterations* , **Inverse Problems** 15:309–327, 1999.
- A. Rieder. *Inexact Newton regularization using conjugate gradients as inner iteration.* **SIAM J. Numer. Anal.** 43:604–622, 2005.

convergence of Newton-type methods with additive random noise

Assumptions:

- (Generalized) IRGNM with iterated Tikhonov regularization.
- source condition with an index function φ which covers $\sqrt{t}$.
- F' Lipschitz continuous (no other nonlinearity condition).
- exponential inequality for the stochastic noise (satisfied e.g. for Gaussian errors)

Results:

- With a-priori stopping rule: Optimal rates of convergence
- With Lepskiĭ balancing principle: Optimal rates up to a $|\log(\sigma)|$ factor.

F. Bauer, T. Hohage, and A. Munk. *Iteratively regularized Gauss-Newton method for nonlinear inverse problems with random noise.* **SIAM J. Numer. Anal.** 47(:1827–1846, 2009.

Outline

1. regularization of nonlinear inverse problems in Hilbert spaces
2. regularization in Banach spaces
3. iterative regularization methods at work
   - inverse obstacle scattering
   - parallel MRI
   - phase retrieval
   - 4Pi microscopy

difficulties in studying inverse problems in Banach spaces

difficulties:

- For a linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces, T' maps from $\mathcal{Y}'$ to $\mathcal{X}'$, not from $\mathcal{Y}$ to $\mathcal{X}$.
- No spectral theorem available.
- How can we formulate algorithms, source conditions?

techniques:

- Use variational rather than spectral methods!

general variational regularization

Tikhonov-type methods:

$$f_\alpha^\delta := \operatorname{argmin}_{f \in D(F)} \left[\mathcal{S}(F(f); Y) + \alpha \mathcal{R}(f)\right]$$

Newton-type methods:

$$f_{k+1}^\delta := \operatorname{argmin}_{f \in D(F)} \left[\mathcal{S}(F(f_k^\delta) + F'[f_k^\delta](f - f_k^\delta); Y) + \alpha_k \mathcal{R}(f)\right]$$

- $\mathcal{S}(\cdot; Y)$: data misfit functional, often negative log-likelihood
- $\mathcal{R}(\cdot)$: penalty term, often $\|\cdot - f_0\|^p$
- Both are usually convex. Hence, in Newton-type methods we have to solve a convex minimization problem in each iteration step.

Why use Banach spaces?

- Use a-priori information on the solution, e.g. sparsity or sharp edges (choice of $\mathcal{R}$)
- Use a-priori information on distribution of data. Gaussian distributions lead to (possibly weigthed) L^2-norms. But one often encounters other distributions, e.g. Poisson, Rice, or salt-and-pepper.

important example: Poisson distributed data

- A random variable Y with values in $\{0, 1, 2, \dots\}$ is called Poisson distributed with parameter $g \geq 0$ if

$$\mathbb{P}(\{Y = y\}) = e^{-g} \frac{g^y}{y!}.$$

- In this case $\mathbb{E}Y = \mathbf{Var}\, Y = g$.
- signal-to-noise ratio: $\frac{\mathbb{E}Y}{\sqrt{\mathbf{Var}\, Y}} = \sqrt{\mathbb{E}Y} = \sqrt{g}$.
- negative log-likelihood:

$$-\ln P_g(Y = y) = -\ln\left(e^{-g} \frac{g^y}{y!}\right) = g - y \ln g + c$$

where c is independent of g.

data misfit functional for Poisson data

- continuous log-likelihood: $\mathcal{S}(g;Y) := \infty$ if not $g \geq 0$, else

$$\mathcal{S}(g;Y) := \int [g - Y \ln g] \, dx$$

- Note: $\mathcal{S}(\cdot;Y)$ convex and $\inf_g \mathcal{S}(g;Y) = \mathcal{S}(Y;Y)$.
- normalization: Set $Y = g^\dagger = F(f^\dagger)$ and $\mathcal{T}(g;g^\dagger) := \mathcal{S}(g;g^\dagger) - \mathcal{S}(g^\dagger;g^\dagger)$ such that $\mathcal{T}(g^\dagger;g^\dagger) = 0$:

$$\mathcal{T}(g;g^\dagger) := \int \left[g - g^\dagger \ln \frac{g}{g^\dagger} - g^\dagger \right] dx$$

$\mathcal{T}$ is called Kullback-Leibler divergence.

assumptions

- $\mathcal{T}\left(\cdot; g^\dagger\right) : \mathcal{Y} \to [0, \infty]$ and $\mathcal{S}\left(\cdot; g^\dagger\right) : \mathcal{Y} \to \mathbb{R}$ convex
- $\mathcal{T}\left(g^\dagger; g^\dagger\right) = 0$.
- There exists $\mathfrak{s} : \mathcal{Y} \to \mathbb{R}$ and $C_{\mathbf{err}}, \mathbf{err} > 0$ such that $\mathcal{S}_{g^\dagger}(g; g^\delta) := \mathcal{S}(g; g^\delta) + \mathfrak{s}(g^\dagger)$ satisfies

$$\begin{aligned} \mathcal{S}_{g^\dagger}\left(F(f); Y\right) &\leq C_{\mathbf{err}} \mathcal{T}\left(F(f); g^\dagger\right) + C_{\mathbf{err}} \mathbf{err} \\ \mathcal{T}\left(F(f); g^\dagger\right) &\leq C_{\mathbf{err}} \mathcal{S}_{g^\dagger}\left(F(f); Y\right) + C_{\mathbf{err}} \mathbf{err} \end{aligned}$$

examples:

- $\mathcal{S}(g_1, g_2) = \mathcal{T}(g_1, g_2) = \|g_1 - g_2\|_{\mathcal{Y}}^p$: $C_{\mathbf{err}} = 2^{p-1}$ and $\mathfrak{s} \equiv 0$
- $\mathcal{S}, \mathcal{T}$ for Poisson data as above: $C_{\mathbf{err}} = 1$, $\mathbf{err} = C_V/t$, and $\mathfrak{s}(g^\dagger) = \int [-g^\dagger + g^\dagger \ln g^\dagger]\, dx$ if $F(f) \geq 0$ and $\int g^\dagger F(f)^2\, dx \leq C_V$ for all f, and $\int \frac{|Y - g^\dagger|^2}{g^\dagger}\, dx \leq \frac{1}{t}$.

Bregman distances

Definition

Let V be a Banach space, $\mathcal{R} : V \to \mathbb{R} \cup \{\infty\}$ convex, $u \in \mathcal{X}$, and $u^* \in \partial\mathcal{R}(u)$. Then

$$D_{\mathcal{R}}^{u^*}(v,u) := \mathcal{R}(v) - \mathcal{R}(u) - \langle u^*, v-u \rangle\,, \qquad v \in V$$

is called the Bregman distance of $\mathcal{R}$ at u and v.

properties:

- $D_{\mathcal{R}}^{u^*}(v,u) \geq 0$, and $D_{\mathcal{R}}^{u^*}(u,u) = 0$.
- If $\mathcal{R}$ is strictly convex, then $D_{\mathcal{R}}^{u^*}(v,u) = 0$ implies $v = u$.
- If V is a Hilbert space and $\mathcal{R}(u) = \|u\|^2$, then $D_{\mathcal{R}}(u,v) = \|u-v\|^2$.
- In general, $D_{\mathcal{R}}$ and $D_{\mathcal{R}}^{\mathrm{symm}}$ do not satisfy the triangle ineq.

variational source conditions

Assume that $D(F)$ is convex and let $f^* \in \partial\mathcal{R}\left(f^\dagger\right) \subset \mathcal{X}'$.

additive form:

$$\forall f \in D(F): \qquad \left\langle f^*, f^\dagger - f \right\rangle \leq \beta_1 \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) + \beta_2 \varphi\left(\mathcal{T}\left(F\left(f\right); F(f^\dagger)\right)\right)$$

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

multiplicative form:

$$\forall f \in D(F): \qquad \left\langle f^*, f^\dagger - f \right\rangle \leq \beta \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right)^{\frac{1}{2}} \varphi\left(\frac{\mathcal{T}\left(F\left(f\right); F(f^\dagger)\right)}{\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right)}\right)$$

B. Kaltenbacher and B. Hofmann. *Convergence rates for the Iteratively Regularized Gauss-Newton Method in Banach Spaces*. **Inverse Problems** 26:035007, 2010.

classical imply multiplicative variational source conditions

Assume that

- $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces
- The following classical source condition is satisfied:

$$f^\dagger - f_0 = \varphi\left(F'\left[f^\dagger\right]^* F'\left[f^\dagger\right]\right) w$$

- $(\varphi^2)^{-1}$ is convex.
- Choose $\mathcal{R}(f) := \frac{1}{2}\|f - f_0\|_{\mathcal{X}}^2$ such that $f^* = f^\dagger - f_0$

Then by Jensen's inequality

$$\begin{aligned}\left\langle f^*, f - f^\dagger\right\rangle &= \left\langle w, \varphi\left(F'\left[f^\dagger\right]^* F'\left[f^\dagger\right]\right)(f - f^\dagger)\right\rangle \\ &\leq \|w\| \left\|f - f^\dagger\right\| \varphi\left(\frac{\left\|F'\left[f^\dagger\right](f - f^\dagger)\right\|^2}{\|f - f^\dagger\|^2}\right).\end{aligned}$$

discussion, open questions

- For additive variational source conditions it is only known in the Hölder case that they follow from classical Hölder source conditions.
- No optimality results for general Banach spaces.
- For additive variational source conditions which are not Hölder, the optimality is even unclear in Hilbert spaces.
- Interpretation of Bregman distances in l^1/L^1 or BV spaces

convergence of Newton-type methods:

Assumptions:

- $f^\delta_{k+1} :=$
  $\mathrm{argmin}_{f\in D(F)} \left[\mathcal{S}(F(f^\delta_k) + F'[f^\delta_k](f - f^\delta_k); Y) + \alpha_k \mathcal{R}(f)\right]$
- Assumptions as explained above plus a variant of the tangential cone condition.

Convergence results:

- Rates, which are optimal in a Hilbert space setting for multiplicative variational source conditions with a-priori stopping rule
- Under additive variational source conditions: Rates with Lepskiĭ stopping rule, but optimality only known for Hölder and Hilbert space setting

T. Hohage, F. Werner. *Iteratively regularized Newton methods for general data misfit functionals and applications to Poisson data.* unpublished manuscript.

further references

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Outline

inverse obstacle scattering problem

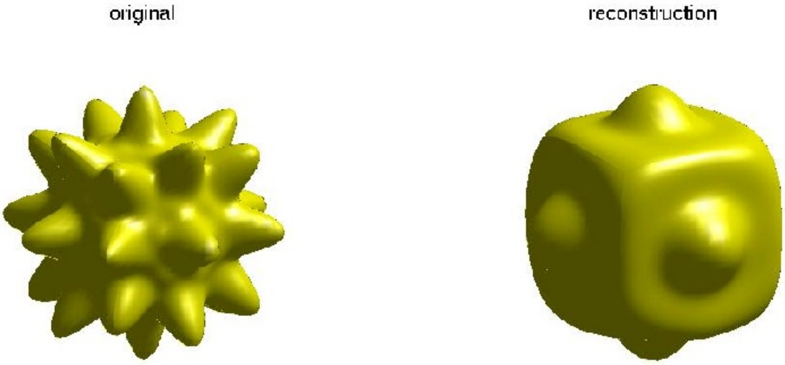
$\Omega \subset \mathbb{R}^m$ compact obstacle, $\mathbb{R}^m \setminus \Omega$ connected
$u_i(x) = e^{-ikx\cdot d}$ incident plane wave with direction $|d| = 1$
$k \approx (\operatorname{diam}\Omega)^{-1}$ wave number

forward problem: Given the domain Ω and the incident field u_i find scattered field u_s such that the total field $u := u_i + u_s$ satisfies

1. the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^m \setminus \Omega$
2. the Sommerfeld radiation condtion $r^{\frac{m-1}{2}} \left(\frac{\partial u_s}{\partial r} - iku_s\right) \xrightarrow{r\to\infty} 0$
3. Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$

inverse problem: Given the far-field pattern u_∞ of u_s, find the shape of Ω!

6 incident waves, wave number k=1

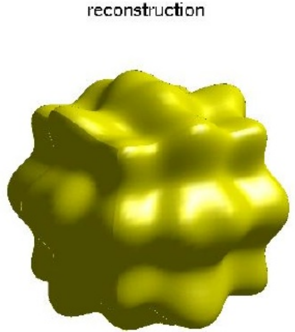


noise level: 5 %

geometry represented by spherical harmonics of degree ≤ 20

H. Harbrecht, T. Hohage. *Fast methods for Three-Dimensional Inverse Obstacle Scattering Problems.* **J. Int. Eq. Appl.** 19:237-260, 2007.

6 incident waves, wave number k=2



noise level: 5 %

geometry represented by spherical harmonics of degree ≤ 20

6 incident waves, wave number k=4

noise level: 5 %

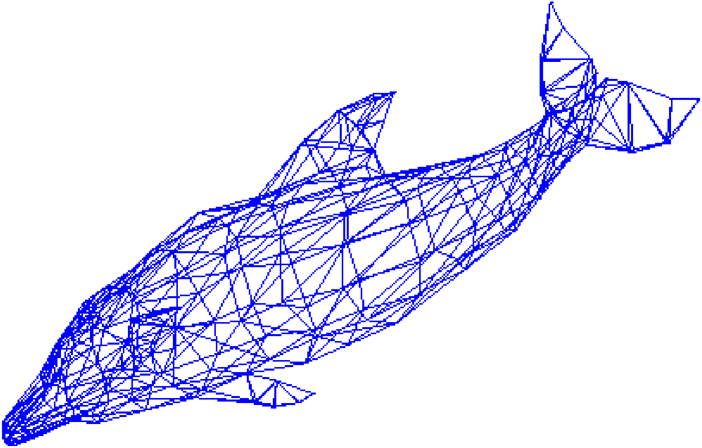
geometry represented by spherical harmonics of degree ≤ 20

1 incident wave from top, wave number k=4

noise level: 2 %

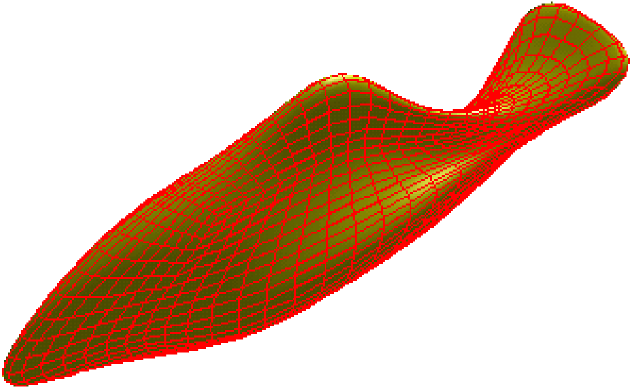
geometry represented by spherical harmonics of degree ≤ 20

a non-star-shaped obstacle



length of dolphin ≈ 2, wave number $k = 8$

reconstruction



6 incident waves, $k = 8$, 1% noise
spherical harmonics of degree ≤ 20

Parallel Magnetic Resonance Imaging (MRI)

Under ideal circumstances, an MRI image can be obtained from the data by inverse Fourier transform. However, there are several reasons for speeding up the measurement process by using several coils in parallel, each of which samples only part of the Fourier space:

- Imaging of moving organs.
- Movements of the patient
- higher efficiency of the instruments

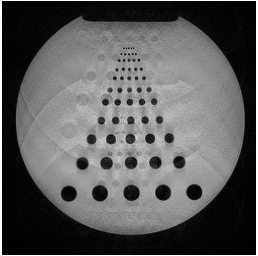
M. Uecker, T. Hohage, K.T. Block, J. Frahm. *Image Reconstruction by Regularized Nonlinear Inversion - Joint Estimation of Coil Sensitivities and Image Content*, **Magnetic Resonance in Medicine** 60:674–682, 2008.

mathematical model

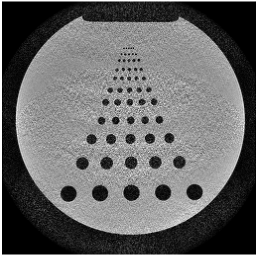
Let $a = a(x, y)$ denote the unknown image (which is complex-valued for biological tissue), and let $c_1, \ldots, c_N$ denote the coil profiles, which are slowly varying complex-valued functions. Moreover, let $\Omega \subset \mathbb{R}^2$ denote the region sampled in Fourier space. Then the forward solution operator is given by

$$F \begin{pmatrix} a \\ c_1 \\ \vdots \\ c_N \end{pmatrix} := \begin{pmatrix} \mathcal{F}(a \cdot c_1)|_\Omega \\ \vdots \\ \mathcal{F}(a \cdot c_N)|_\Omega \end{pmatrix}.$$

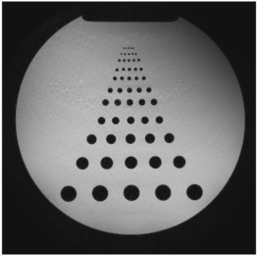
parallel MRI reconstructions



GRAPPA

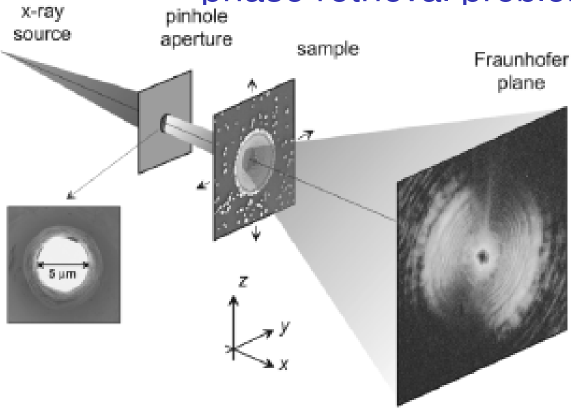


IRGNM



IRGNM with real constraint

phase retrieval problems



Coherent x-ray source allow high resolution even with simple pinhole experiments.

Pauli problem: Reconstruct a function $u_0 \in L^2(\mathbb{R}^d)$ given noisy observations of

$$F(u_0) := |\mathcal{F}_d u_0|^2$$

and additional a-priori information on u_0.

Fresnel and Fraunhofer approximations

- solution to Helmholtz equation in half-space $\{\mathbf{x} : x_3 \geq 0\}$:

$$u(\mathbf{x}', x_3) = \mathcal{F}_2^{-1} e^{ix_3\sqrt{k^2-|\xi'|^2}} \mathcal{F}_2 u_0, \qquad u_0 = u(\cdot, 0)$$

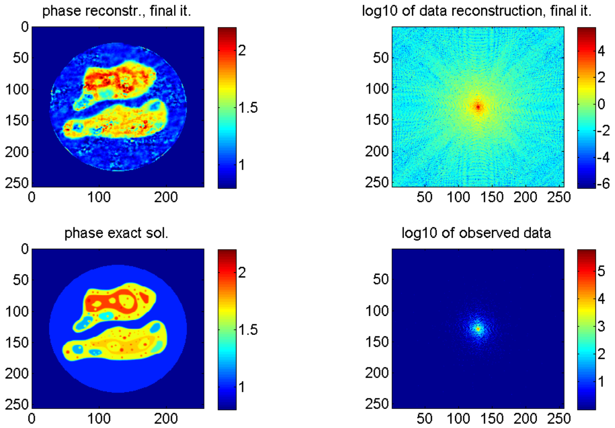
- Fresnel approximation: $\sqrt{k^2-|\xi'|^2} \approx k - \frac{|\xi'|^2}{2k}$ if $|\xi'| \ll k$

$$\begin{aligned} u(\mathbf{x}', x_3) &\approx e^{ikx_3} \mathcal{F}_2^{-1} e^{\frac{-ix_3|\xi'|^2}{2k}} \mathcal{F}_2 u_0 \\ &= \frac{ik}{2\pi x_3} e^{ikx_3 + \frac{ik}{2x_3}|\mathbf{x}'|^2} \int_{\mathbb{R}^2} e^{\frac{ik|\mathbf{y}'|^2}{2x_3} - \frac{ik}{x_3}(\mathbf{x}' \cdot \mathbf{y}')} u_0(\mathbf{y}') d\mathbf{y}' \end{aligned}$$

- Fraunhofer approximation: $k|\mathbf{y}'|^2 \ll x_3$

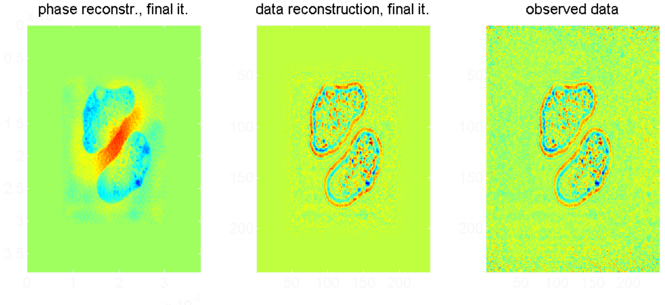
$$\begin{aligned} u(\mathbf{x}', x_3) &\approx \frac{ik}{2\pi x_3} e^{ikx_3 + \frac{ik}{2x_3}|\mathbf{x}'|^2} \left[\mathcal{F}_2 u_0 \left(\frac{\bullet}{x_3}\right)\right](\mathbf{x}') \\ |u(\mathbf{x}', x_3)| &\approx \frac{k}{2\pi x_3} \left| \left[\mathcal{F}_2 u_0 \left(\frac{\bullet}{x_3}\right)\right](\mathbf{x}') \right| \end{aligned}$$

reconstruction of a phase object from simulated Fraunhofer data



a-priori information: $u_0 = \exp(if)$ with f real-valued and $\operatorname{supp} f =$ known disk.

reconstruction of a cell from holographic experimental data in the Fresnel regime

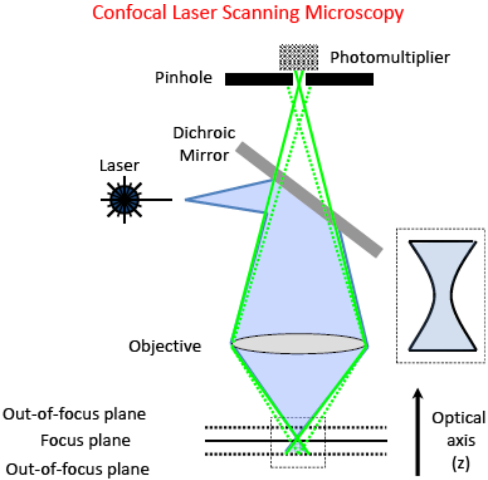


experimental data published in:

K. Giewekemeyer, S.P. Krüger, S. Kalbfleisch, M. Bartels, C. Beta, T. Salditt.
X-ray propagation microscopy of biological cells using waveguides as a quasipoint source. **Phys. Rev. A** 83:023804. 2011

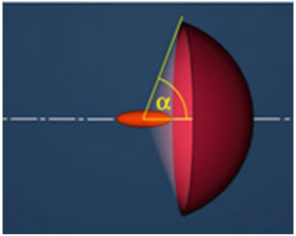
confocal fluorescence microscopy

- Focused light excites fluorescent markers in the object.
- 3d imaging of living cells
- leads to 3d deconvolution problem with nonnegativity constraints and Poisson data.



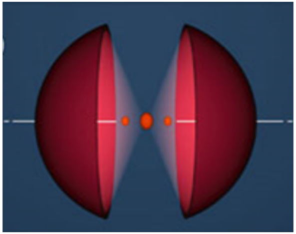
4Pi Microscopy

lateral res. > 200nm



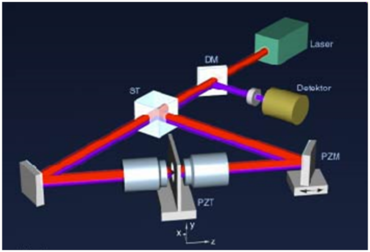
axis resol. > 500nm

lateral res. > 200nm



axis resol. $> 70 - 140$nm

Experimental setup:



S. Hell and E. Stelzer,
J. Opt. Soc.Am.
A9:2159–2166
(1992).

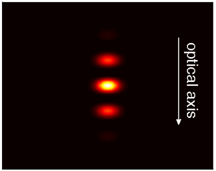
4Pi point spread function

psf model for 4Pi microscopy of type A:

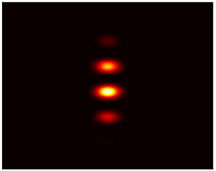
$$h_{4\mathrm{Pi}}^{\phi}(\mathbf{x}) \approx h_{\mathrm{confocal}}(\mathbf{x})\cos^{\gamma}(\beta\tilde{z}+\phi)$$

$\phi =$ relative phase of interfering photons
$\gamma \in \{2,4\}$, $\beta > 0$ a-priori known by experimental setup

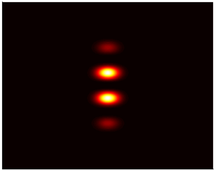
S. Hell and A. Schönle. in Science of Microscopy. p. 790–834, Springer, New York (2006).



$\phi = 0$

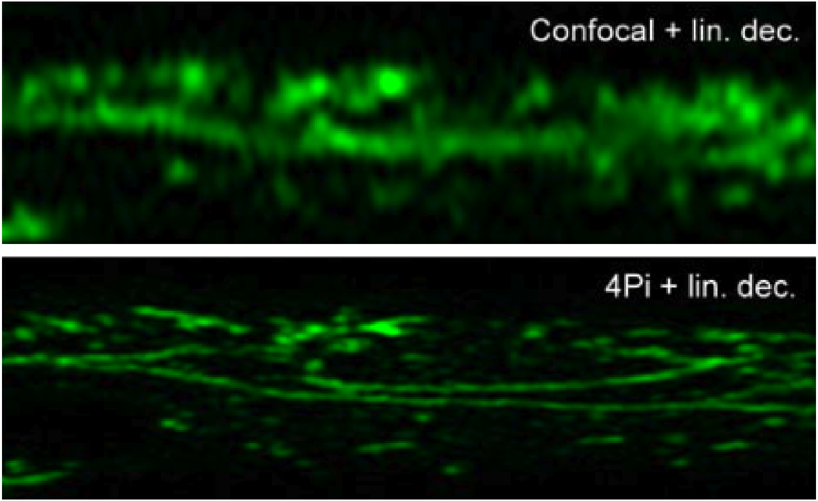


$\phi = \frac{\pi}{4}$



$\phi = \frac{\pi}{2}$

confocal vs 4Pi



microtubular network in a PtK-2 cell

M.C. Lang, T. Staudt, J. Engelhardt, S.W. Hell, **New Journal of Physics** 10 (2008)

joint estimation of object and phase

Main difficulty:

- The phase ϕ varies (slowly) in space due to variations of the refractive index.
- $\phi(x)$ has to be estimated together with $f(x)$.

operator formulation of the inverse problem:

$$F(f, \phi) = g \qquad \text{s.t.} f \geq 0$$

with

$$(F(f, \phi))(\mathbf{x}) := \int \mathbf{h}(\mathbf{y} - \mathbf{x}, \phi(\mathbf{x}))\mathbf{f}(\mathbf{y})\, \mathbf{dy}$$

- R. Stück. *Semi-blind deconvolution in 4Pi-microscopy.* PhD theis, University Göttingen, 2011.

microtubules in Vero cells