INSTITUTO DE CIENCIAS MATEMÁTICAS

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MATERIAL GEOMETRY

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Dedicado a quien está por llegar.
Por siempre, el mayor de mis logros.

Leibniz wrote,
"His paucis consideratis, tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum."

"These few things having been considered, the whole matter is reduced to pure geometry, which is the one aim of physics and mechanics."

Contents

Scientific articles of the thesis

The theory developed and the results proved which are presented in this work in fulfillment of the thesis requirement for the degree of Doctor of Philosophy is fundamentally contained in a serie of scientific articles which are exposed here:

- V. M. Jiménez, M. de León and M. Epstein. Material distributions. *Mathematics and Mechanics of Solids.* 0(0):1081286517736922,0.
- V. M. Jiménez, M. de León and M. Epstein. Characteristic distribution: An application to material bodies. *Journal of Geometry and Physics*, 127:19-31, 2018.
- V. M. Jiménez, M. de León and M. Epstein. Lie groupoids and algebroids applied to the study of uniformity and homogeneity of cosserat media. *International Journal of Geometric Methods in Modern Physics*, 15(08):1830003, 2018.
- M. Epstein, V. M. Jiménez and M. de León. Material geometry. *Journal of Elasticity*, 135 (1): 237-260, 2019.
- V. M. Jiménez, M. de León and M. Epstein. Lie groupoids and algebroids applied to the study of uniformity and homogeneity of material bodies. *Journal of Geometric Mechanics*, 11(3),301-324 2019.

- V. M. Jiménez, M. de León and M. Epstein. On the homogeneity of non-uniform material bodies. *Accepted in a volume for Springer/Birkhauser on Geometric Continuum Mechanics*, 2019.

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A note to the reader

We would like to remember that this memory has been written with the hope of being as easy as possible to read. The treatment is essentially self-contained and proofs are complete. The prerequisites essentially consist in a working knowledge of basic notions of *differetial geometry*.
So, the necessary fundamental and non-elementary concepts has being presented inside the chapter 2 in such a way that it could serve as an independent introduction to the notions. **However, the reader who is already familiarized with these notions does not need to read them to understand the development made in the thesis.**

Abstract

In *continuum physics* the physical properties of a elastic body are characterized for all the constitutive relations. This measures the mechanical response produced at each particle by a deformation in a local neighbourhood of the particle. Differential geometry provides a rigorous mathematical framework not only to present the constitutive properties but to discover and prove results. For applications, it is usual that the bodies are assumed uniform and homogeneous in the sense of that the body is made of a unique material and there is a configuration in such a way that the mechanical response is the same at all the points.
The main purpose of this thesis is to follow the Noll's approach to present a mathematical framework based on groupoids, algebroids and distributions to deal with non-uniform and inhomogeneous simple bodies.
For any simple body a unique groupoid, called *material groupoid*, may be naturally associated. The unifomity of the body coincides with the transitivity of the groupoid. If the material groupoid turns out to be a Lie groupoid the associated Lie algebroid, called *material algebroid*, is available. Then, the homogoneneity is characterized by the integrability of both (material groupoid and material algebroid).
However, the property of being Lie groupoid is not guaranteed. In fact, smooth uniformity corresponds to that imposition of differentiability on the material groupoid. Smooth distributions are now introduced to deal with this case. In fact, two smooth distributions, called *material distributions*, may be canonically defined generalizing the notion of Lie algebroid. Thus, it is proved that we can cover the simple body by a material foliation whose

leaves are (smoothly) uniform. These new tools are also used to present a "measure" of uniformity and an homogeneity for non-uniform bodies.
The construction of the material distribution is generalized to a much more abstract framework in which the case of an arbitrary subgroupoid of a Lie groupoid is treated. We also study Cosserat media by imposing that the corresponding material groupoid is a Lie groupoid.

Resumen

En *Física de medios continuos* las propiedades de un cuerpo elástico están caracterizadas por todas las relaciones constitutivas. Esto mide la respuesta mecánica producida en cada partícula por una deformación en un entorno local de la misma. La geometría diferencial proporciona un marco matemático riguroso no sólo para presentar las propiedades constitutivas, sino para descubrir y probar nuevos resultados. En cuanto a aplicaciones, es usual imponer que los cuerpos sean uniformes y homogéneos en el sentido de que un cuerpo está hecho de un único material y hay una configuración de tal manera que la respuesta mecánica es la misma en todos los puntos.
El objetivo principal de esta tesis es seguir el enfoque de Noll para presentar a marco matemático basado en grupoides, algebroides y distribuciones con el objetivo de tratar con cuerpos simples inhomogeneos no uniformes.
Cualquier cuerpo simple tiene asociado de manera natural un único grupoide, llamado *grupoide material.* La uniformidad del cuerpo coincide con la transitividad del grupoide. Si el material resulta ser un grupoide de Lie, el algebroide asociado, llamado *algebroide material*, está disponible. Entonces, la homogeneidad es caracterizada por la integrabilidad de ambos (el grupoide material y el algebroide material).
Sin embargo, la propiedad de ser un grupoide de Lie no está garantizada. De hecho, la uniformidad diferenciable corresponde a esta imposición de diferenciabilidad sobre el grupoide. Introducimos ahora las distribuciones diferenciables para tratar con este caso. Así, dos distribuciones diferenciables, llamadas *distribuciones materiales*, pueden ser canónicamente definidas generalizando la noción de algebroide de Lie.

Se prueba con esto que todo cuerpo simple puede ser cubierto por una foliación material cuyas hojas son (diferenciablemente) uniformes. Estas nuevas herramientas son también usadas para presentar una "medida" de uniformidad y una homogeneidad para cuerpos no uniformes.
La construcción de la distribución material es generalizada a un marco mucho más abstracto en que se trata el caso de un subgrupoide (no necesariamente subgrupoide de Lie) arbitrario de un grupoide de Lie. Se estudian también medios de Cosserat imponiendo que el correspondiente grupoide material sea un grupoide de Lie.

Chapter 1

Introduction

From its origins, theory of elasticity has been a rich and exciting branch of mathematical research. This subject was initially created by J. Bernoulli, A. L. Cauchy and L. Euler and and there is a long list of important mathematicians who made some substantial contribution to this branch: Beltrami, Birkhoff, Hadamard, Lipschitz among others.
Over the years, however, the importance of elasticity as a branch of mathematics was decreasing. Although there were still exceptions, in general mathematicians lost the intererest in elasticity (see [93]).
The rebirth may be dated in 1954 with the thesis of W. Noll entitled "*On the Continuity of the Solid and Fluid States*" [75]. Here, W. Noll started to use the concept of *material points*. This subject, which could be called *new rational elasticity*, is the physical basis of this memory.
In particular, we will be interested in the interaction between Continuum Mechanics and Differential Geometry. As may be found in the modern books *Introduction to rational elascity*, due to C. C. Wang and C. Truesdell, and *Mathematical foundations of elasticity*, due to J. E. Marsden and T. J. R. Hughes, this relation has produced in a very rich theory full of interesting results. It is remarkable that this relationship has even older history. In fact, theories of elastic beams and shells had already needed the use of results of differential geometry of curves and surfaces. The work of Cosserat

brothers anticipated certain aspects of modern differential geometry by the adding of microstructure to the material body.

The clearest link between these two areas is provided by the fact of that a "*continuum*" is physically modelized as a 3−dimensional connected manifold $\mathcal{B}$, the *material body*, which can be embedded in $\mathbb{R}^3$, i.e., $\mathcal{B}$ has a global chart. The physical space is identified with $\mathbb{R}^3$. Observe that a material body is, by definition, an abstract topological space. Then, as a matter of applications, we need to depict the body into the physical world. To deal with this problem, we define the *configurations*. A configuration ϕ is just an embedding from $\mathcal{B}$ to $\mathbb{R}^3$. We usually fix a reference configuration ϕ_0. Then, a *deformation* is simply a change of configurations $\phi \circ \phi_0^{-1}$.

The modern formulation [85] has achieved to show that we may remove most of the limitations without affecting the physical part of the theory. In fact, in [85] a rigorous theory arises from imitating the geometric approach of Classical Mechanics by stating that the configuration space is, forgive the redundancy, the space of configurations. Notice that the space of configurations has a (non-unique) structure of infinite-dimensional differentiable manifold [49, 58].

In this thesis we will work on another facet of the interaction between Continuum Mechanics and Differential Geometry. On the one hand, the theory describing the elastic fields of certain kind of defects, which are now called *dislocations* and *disclinations*, was originally developed by Vito Volterra in 1907 [90]. However, it was not until 1955 when a rigorous theory of continuous distributions was conceived by K. Kondo [57], D. A. Bilby [4], E. Kröner [59] [60], J.D. Eshelby [45] and others (see also the books [61, 74]). This structurally based theory is motivated by heuristic considerations, mostly studying limiting process starting from a defective crystalline structure.

There is another distinct approach proposed by W. Noll [76]. Although both approaches use similar geometric structures, the conceptual status of the theory, however, is really different. In fact, the Noll's approach is based on the existence of *constitutive laws* encoding all the information about the material response of the body. This permits us to compare material points via the so-called *material isomorphisms* (notion which will be discussed below). Being some of the results achieved by this school of thought the same as those of its predecessors, we can even find important differences. In fact, the fundamental role of the *material symmetry groups* in the Noll's

theory in one of them.
The notion of material isomorphism is the main idea of the Noll's theory. Denote by W the mechanical response of the body $\mathcal{B}$, such as the *Cauchy stress*, or the elastic energy per unit mass. Let us assume that $\mathcal{B}$ is a *simple elastic body*, i.e., the constitutive laws are completely characterized at a point by the point and the infinitesimal deformation at the same point. Then, the mechanical response is represented as a differentiable map $W : \mathcal{B} \times Gl\left(3, \mathbb{R}\right) \to V$ (where $Gl\left(3, \mathbb{R}\right)$ is general linear group of regular $3 \times 3-$matrices and V is a finite dimensional vector space). Notice that the form of W depends on the choice of a particular reference configuration ϕ_0 to express the gradient of the deformations.
Let $X, Y \in \mathcal{B}$ be two particles. X and Y are said to be *materially isomorphic* if there exists a linear isomorphism $P_{XY} : T_X\mathcal{B} \to T_Y\mathcal{B}$ such that

$$W(F, Y) = W(FP_{XY}, X), \tag{1.1}$$

for all deformation gradients F. In this case P_{XY} is called *material isomorphism.* If $X = Y$, P_{XX} is called *material symmetry.* The idea of materially isomorphic body points comes to endow of a mathematical rigor to the physical property of being made of the same material.
A body $\mathcal{B}$ is *materially uniform* if there are no two different materials in $\mathcal{B}$ or, equivalently, if all its points are materially isomorphic. So, we may claim that two body points X and Y are materially isomorphic if, and only if, their constitutive functions belong to the same orbit under the action mentioned above.
Material isomorphisms P_{XY} are not, in general, unique. In fact, the set of material isomorphisms $G\left(X, Y\right)$ from X to Y may only satisfy one of these two conditions

- $G\left(X, Y\right) = \emptyset$.
- $G\left(X, Y\right)$ is in a one-to-one correspondence with the material symmetry group $G\left(X\right)$ at X. In fact,

  $$P_{XY} \cdot G\left(X\right) = G\left(X, Y\right),$$

  for all material isomorphism $P_{XY} \in G\left(X, Y\right)$.

Taking into account these facts and assuming that the body is *smoothly uniform* Noll introduced the notion of *material parallelisms* and their associated curvature-free *material connections*. This was further extended by T. J. Wang [92] and Bloom [6]. We present here [54] another approach to these notions.

The non-vanishing of the torsion of the (non necessarily unique) material connections measures the presence of defects of the material. In the case of dicrete symmetry groups, the material connection is unique and, then, we have a "*canonical*" tensor (torsion) measuring the defects. In Noll's terminology, the notion of *local homogeneity* of the body is physically interpreted as the absence of defects.

The frame bundle of the body and, more particularly, its $G-$structures provides a new formulation of these ideas. In [31] M. Elżanowski, M. Epstein and J. Śniatycki associate to any smoothly uniform body a family of conjugated $G-$structures, *material $G-$structures*, in such a way that their flatness characterizes the homogeneity of the material.

Here, we face the case of non-uniform bodies by using the theory of *Groupoids*. In fact, the history of this thesis begins with the knowledge of that the collection of all material isomorphisms P_{XY} for all pairs of body points X, Y of a material body $\mathcal{B}$ is a groupoid, called *material groupoid*, which is a subgroupoid of the $1-$jets groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$ on $\mathcal{B}$ ([36, 37]). The material groupoid of $\mathcal{B}$ will be denoted by $\Omega(\mathcal{B})$. Therefore, uniformity and homogeneity will be studied by using the properties of the material groupoid.

As a first consequence of the structure of groupoid of $\Omega(\mathcal{B})$: $\mathcal{B}$ is uniform if, and only if, $\Omega(\mathcal{B})$ is transitive. If the material groupoid $\Omega(\mathcal{B})$ a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$, the *associated Lie algebroid* $A\Omega(\mathcal{B})$ is available. Therefore, a serie of results are presented characterizing the homogeneity by the properties of the Lie algebroid [54].

A natural question now arises: Is always $\Omega(\mathcal{B})$ a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$? Actually, the answer is negative. In fact, we proved that $\mathcal{B}$ is smoothly uniform if, and only if, $\Omega(\mathcal{B})$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. We also gave particular examples of non-uniform bodies in which the material groupoid is not a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.

Thus, we should face the case in which $\Omega(\mathcal{B})$ is simply an algebraic subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. Even in that case, we may generalize the construction of the associated Lie algebroid to construct a smooth distribution of $\Pi^1(\mathcal{B},\mathcal{B})$ called *material distribution* $A\Omega^T(\mathcal{B})$ (see [50] or [51]). $A\Omega^T(\mathcal{B})$ is generated by the (local) left-invariant vector fields on $\Pi^1(\mathcal{B},\mathcal{B})$ which are in the kernel of TW. Due to the groupoid structure, we can still associate two new objects to $A\Omega^T(\mathcal{B})$, denoted by $A\Omega(\mathcal{B})$ and $A\Omega^\sharp(\mathcal{B})$, as defined by the following diagram:

$$\begin{array}{ccc}
\Pi^1(\mathcal{B},\mathcal{B}) & \xrightarrow{A\Omega^T(\mathcal{B})} & \mathcal{P}\left(T\Pi^1(\mathcal{B},\mathcal{B})\right) \\
\Big\uparrow{\scriptstyle \epsilon} & \overset{A\Omega(\mathcal{B})}{\nearrow} & \Big\downarrow{\scriptstyle T\alpha} \\
\mathcal{B} & \xrightarrow{A\Omega^\sharp(\mathcal{B})} & \mathcal{P}(T\mathcal{B})
\end{array}$$

Here $\mathcal{P}(E)$ defines the power set of E, $\epsilon(X)$ is the identity map of $T_X\mathcal{B}$ and $\alpha : \Pi^1(\mathcal{B},\mathcal{B}) \to \mathcal{B}$ denotes the source map of the groupoid.

By construction, the distributions $A\Omega^T(\mathcal{B})$ and $A\Omega^\sharp(\mathcal{B})$, are integrable (in the sense of Stefan [86] and Sussmann [88]), and they provide two foliations, $\overline{\mathcal{F}}$ on $\Pi^1(\mathcal{B},\mathcal{B})$ and $\mathcal{F}$ on $\mathcal{B}$, such that $\Omega(\mathcal{B})$ is union of leaves of $\overline{\mathcal{F}}$. As a consequence, we have that $\mathcal{B}$ can be covered by a foliation of some kind of smoothly uniform "sub-bodies", called *material submanifolds*. The material distribution also offers a tool apt to provide a general classification of smoothly non-uniform bodies and the possibility to distinguish various degrees of uniformity. In addition, homogeneity may be generalized in such a way that any simple body can be tested to be homogeneous.
Next, we may consider a more general situation. We study the problem from a purely mathematical framework, since we are convinced that this analysis should be relevant not only for its applications to Continuum Mechanics, but also for the general theory of groupoids.

So, let $\overline{\Gamma} \subseteq \Gamma$ be a subgroupoid of a Lie groupoid $\Gamma \rightrightarrows M$; notice that we are not assuming, in principle, any differentiable structure on $\overline{\Gamma}$. Even in that case, we can construct a generalized distribution $A\overline{\Gamma}^T$ over Γ generated by the (local) left-invariant vector fields on Γ whose flow at the identities is totally contained in $\overline{\Gamma}$. This distribution $A\overline{\Gamma}^T$ will be called

the *characteristic distribution of* $\overline{\Gamma}$. Again, due to the groupoid structure, we can still associate two new objects to $A\overline{\Gamma}^T$, denoted by $A\overline{\Gamma}$ and $A\overline{\Gamma}^\sharp$, called the *base-characteristic distribution of* $\overline{\Gamma}$, analogously to the above diagram.
The relevant fact is that both distributions, $A\overline{\Gamma}^T$ and $A\overline{\Gamma}^\sharp$, are integrable and they provide two foliations, $\overline{\mathcal{F}}$ on Γ and $\mathcal{F}$ on M. Studying the properties of these foliations we obtain the following two main results:

Theorem 2.34 Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie groupoid) over M. Then, there exists a maximal foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.

Theorem 2.32 For each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}(\mathcal{F}(x))$ of Γ with base $\mathcal{F}(x)$.

So, although our groupoid $\overline{\Gamma}$ is not a Lie subgroupoid of Γ, we can still cover it by manifolds (leaves of the foliation $\overline{\mathcal{F}}$) and extract "transitive" and "differentiable" components (the Lie groupoids $\overline{\Gamma}(\mathcal{F}(x)) \rightrightarrows \mathcal{F}(x)$).

Finally, following the mentioned work Cosserat brothers, there are materials like rocks which cannot be modelled as simple materials [10]. A *Cosserat continuum* arises from adding extra kinematical variables and can be described as a linear frame bundle $F\mathcal{B}$ of a manifold $\mathcal{B}$ which can be covered with just one chart (see [37]). $\mathcal{B}$ is called *macromedium* of *underlying body*.
Then, a *configuration* of $F\mathcal{B}$ is an embedding $\Psi : F\mathcal{B} \to F\mathbb{R}^3$ of principal bundles such that the induced Lie group morphism is the identity map. We fix a configuration Ψ_0, as the *reference configuration*, and a *deformation* is a change of configurations, namely $\kappa = \Psi \circ \Psi_0^{-1}$.
Therefore, the *constitutive elastic law* is now written as

$$W = W\left(X, \overline{F}\right),$$

where X is a point of the macromedium and $\overline{F}$ is the gradient of a deformation κ at a point X. Without going into details, the *material isomorphisms* are again defined as the symmetries by the right action of

matrices of the constitutive law.
The notion of *non-holonomic material groupoid of second order* associated to a Cosserat continuum $F\mathcal{B}$ arises again in a very natural way. Actually, the collection of all the $1-$jets of material isomorphisms constitutes a groupoid, denoted by $\overline{\Omega}(\mathcal{B})$, over the underlying body $\mathcal{B}$. Thus, under the assumption of differentiability over the non-holonomic material groupoid of second order we obtain similar results to those related with simple materials [52].

The thesis may be divided in two parts clearly differentiated and separated by a chapter entitled "Prelude". The first part is confined in chapter 2 (Fundamentals). This part is devoted to present an introduction to the non-elementary knowledge necessary to understand the results given in the thesis. This chapter is divided into three parts, Continuum Mechanics, Groupoids and Algebroids, and each of these parts is only oriented to the reader who is not an expert on the subject. In fact, the goal of this chapter is to make the thesis as self-contained as possible to ease the reader the comprehension of the memory. Those who have a wide knowledge on the topic can skip the correspondent section without missing any contribution of the thesis.

In this way, section 2.1 is the encharged of introducing the reader in the world of Continuum Mechanics. In particular, we will focus on the *Constitutive Theory of Materials* giving a very brief introduction. In this introduction we will familiarize the reader with notions like uniformity and homogeneity which will have a great importance in the thesis. We will present two different kind of materials: Simple materials and Cosserat media.
Groupoids are the topic of the section 2.2. We start the section analyzing a groupoid induced by a real game, *15-puzzle groupoid*, to help the reader to get an idea of what a groupoid is. Then, we present a rigorous theory of groupoids focusing on the results in which we are interested in. We also give several examples of groupoids. Two particularly relevant examples for this thesis will be the following: the 1-jets groupoid and the second-order non-holonomic groupoid. As a last introductory section, section 2.3 is devoted to Lie algebroids. Similarly to the previous section, we start with a simple example of Lie algebroid, the tangent algebroid, to give

an idea of what a Lie algebroid is. Next, the category of Lie algebroid is constructed presenting a consistent theory of Lie algebroid with several examples. An explicit construction of the Lie's functor from Lie groupoids to Lie algebroids is then studied. As examples arise the *1-jets algebroid* and the *second-order non-holonomic algebroid* which will be important in some of the results of the thesis. So, we show a Lie algebroid isomorphism from the 1-jets Lie algebroid to the algebroid of derivations which gives us a different way to depict the 1-jets algebroid. As a final part of this section, we study the Lie's fundamental theorems for Lie groupoids and Lie algebroids. We begin studying Lie's first fundamental theorem, which shows that any integrable Lie algebroid can be integrated by a Lie groupoid with simply connected $\beta-$fibres. Then, with a similar development, it can be proved a theorem about integrability of subalgebroids. On the other hand, we will describe Lie's second fundamental theorem which studies the integration of Lie algebroid morphisms and we will use this to prove consequences which will be useful in the thesis. We conclude giving an example of a non-integrable Lie algebroid to show that the Lie's third fundamental theorem is not true.
As the name indicates, the chapter "Prelude" work as a introduction and a motivation to the study made in the thesis. In particular, we present the material groupoid which is maybe the *cornerstone* of this thesis. We also prove here two results characterizing the (smooth) uniformity over the material groupoid which give us an intuition about the path which we have taken in the thesis.
Now, it begins the second part of the thesis. Except for a few results shown in the "prelude", all the new developments of the thesis are presented hereinafter. This part is divided in two chapters: The first chapter exhibits the results obtained as consequences of assuming that the material groupoid is a Lie subgroupoid of the 1-jets groupoid. In the second part we study the material groupoid in a more general sense (without the assumption of regularity imposed in the previous chapter) introducing new tools to deal with this case.
Thus, in chapter 3 are explained in detail the results presented in [52, 54] although there are new results which have not been published yet. This chapter is separated in two sections. The first section 3.1 is for simple materials. Here we calculate the material algebroid using the algebroid of derivations to give more than one representation of this Lie algebroid. We

use this to characterizes, above all, the homogeneity in different ways. A similar development is made in the next section for Cosserat media [52]. Here the results are rather more sophisticated due to the adding structure of this kind of materials.
Chapter 4 is devoted to show the results published in [39, 50, 51, 53]. Again, there are also non-published results presented in this chapter. We divide this chapter in three sections. First section deals with the general case of a subgroupoid $\overline{\Gamma}$ (not necessarily a Lie subgroupoid) of a Lie groupoid $\Gamma \rightrightarrows M$. So, we construct the so-called characterisic distributions which provides us a "pseudo-differentiable" structure on the subgroupoid $\overline{\Gamma}$ generalizing the structure of Lie groupoid. In particular, we obtain a way to give a "pseudo-differentiable" structure on any subset N of a manifold M generalizing (in some "maximal" way) the structure of smooth manifold. In section 4.2 we apply all the results of the previous section to simple materials. In this case, characteristic distributions will have a particular shape and will be called material distributions. Thus, as an interesting result, we obtain that any body $\mathcal{B}$ can be covered by a maximal foliation of some kind of "(smoothly) uniform subbodies". We also present a "measure of uniformity", the *graded uniformity*, based on the material distributions. Finally, an homogeneity for non-uniform bodies is shown and characterized in different ways. As we predicted, the material groupoid will not have to be a Lie subgroupoid of the 1-jets groupoid. Thus, in section 4.3 we give some examples of this. We study the graded uniformity and the homogeneity on these examples.
Two appendices are presented at the end of the memory. Appendix A is devoted to *Principal bundles.* In particular, the *frame bundle* and the *second-order non-holonomic frame bundle* are introduced. The *integrability* of $G-$structures and second-order non-holonomic $\overline{G}-$structures, which is a fundamental notion in our thesis, is studied. In appendix B we present the concepts of foliations and distributions. Here we prove some classical integrability theorems which will be used in the thesis.

Chapter 2

Fundamentals

This chapter is devoted to introduce the necessary non-elementary fundamental notions to understand the results of this thesis. Here we will give an introduction to *Simple bodies*, *Cosserat media*, *Groupoids* and *Algebroids*.

2.1 Continuum Mechanics

We will start with a very brief sketch of the indispensable background in *Continuum Mechanics*. We will mainly follow the books [33, 38, 93]. Another recommendable reference is [67].

Elastic Simple body

To move from classical mechanics of finite systems of particles to mechanics of continuum materials arises the problem of finding a proper definition of *body*. In this thesis we are interested in the so-called *deformable body*. Such a model defines a *body* as an oriented manifold $\mathcal{B}$ of dimension 3 which can be covered by just one chart. The points of the manifold $\mathcal{B}$ will be called *body points* or *material particles* and will be denoted by using capital letters

$(X, Y, Z \in \mathcal{B})$. A *subbody* of $\mathcal{B}$ is a open subset $\mathcal{U}$ of the body.
It is important to note that, by definition, a material body (and its material particles) is just a topological space. So, one could think that the body lives in some "*abstract world*" outside the reality. To manifest the material inside the "*real world*" there exist the so-called *configurations.*

Definition 2.1.1. A *configuration* of a body $\mathcal{B}$ is given by an embedding $\phi : \mathcal{B} \to \mathbb{R}^3$. The $1-$jet $j^1_{X,\phi(X)}\phi$ at the body point $X \in \mathcal{B}$ (see appendix A) is called an *infinitesimal configuration at* X.

Thus, a configuration of a material body assigns to any particle X a spot in the space in a smooth way such that two particles cannot be assigned to the same spot. Points on the euclidean space $\mathbb{R}^3$ will be called *spatial points* and will be denoted by lower case letters $(x, y, z \in \mathbb{R}^3)$.
We usually identify the body with one of its configurations, say ϕ_0, called *reference configuration.* Coordinates in the reference configuration will be denoted by X^I, while any other coordinates will be denoted by x^i.
In spite of the choice of the reference configuration, any formulation should turn out to be independent of this choice. The established physical rules should not depend on the representation of the body in the real or physical world.
A statement or property on the material body $\mathcal{B}$ is said to be *configuration indifferent* or *configuration independent* if it does not depend on the chosen reference configuration (*uniformity* 2.1.5 will be a good example of configuration independent property).

Definition 2.1.2. Given any arbitrary configuration ϕ, the change of configurations $\kappa = \phi \circ \phi_0^{-1}$ is called a *deformation*, and its $1-$jet $j^1_{\phi_0(X),\phi(X)}\kappa$ is called an *infinitesimal deformation at* $\phi_0(X)$.

Notice that, by using the *Polar decomposition theorem*, any infinitesimal deformation at a body point may be decomposed as follows,

$$F = RU = VR, \tag{2.1}$$

where R is a an orthogonal matrix and U and V are symmetric positive-definite tensors. R is called the *rotation tensor* and U and V the *right* and *left stretch tensors of* F. An important circumstance results when the determinant of F is positive. In that case, the rotation tensor R is a *pure rotation* (no mirror needed). On the other hand, any (real) positive definite matrix may be diagonalized, i.e., it is similar to a diagonal matrix given by its eigenvalues. So, the physical interpretation of Eq. (2.1) is that the effect of a infinitesimal deformation at a point on a vector is to apply three stretches (given by the eigenvalues of U) and a rotation given by R. Analogously we interpret $F = VR$. Observe that, Eq. (2.1) permits us to separate the strain information (U or V) from the rotation information (R).

The *right* and *left Cauchy-Green tensors of* F are given by $C = F^T F = U^2$ and $B = FF^T = V^2$. If two infinitesimal deformations have the same right (or left) stretch tensor, one follows from the other by a rotation.

In term of local coordinates, by following notation introduced above, the entries of the matrix F are denoted as follows,

$$F^i_J = \frac{\partial x^i}{\partial X^J} \tag{2.2}$$

where the transformation is the composition of the local coordinates $\left(x^i\right)$ with the inverse of the reference configuration $\phi_0 = \left(X^I\right)$. Then, the Cauchy-Green tensors have the following shape

$$C_{IJ} = F^i_I F^i_J$$

$$B^{ij} = F^i_I F^j_I$$

The Einstein summation convention will be used along the whole manuscript. There is still another important tensor to introduce: The *Lagrangian strain tensor*. The Lagrangian strain tensor is given by the formula:

$$E = \frac{1}{2}\left(C - I\right) \tag{2.3}$$

where I is the identity matrix (or the identity tensor in the reference configuration).

To be able to predict the deformation of a body in motion is an important goal in continuum mechanics. It is very intuitive that the internal constitution of the body should play a role. For instance, steel, wood or gel will not be deformed equally when they are subject to the same loading.

The mathematical interpretation of this is that the dymanical principles alone should not be enough to determine the motion of a deformable body. In fact, the response of the body to the history of its deformations is supposed to be characterized for one or more *constitutive equations*.
The experiments seem to indicate that the material response is a *local property* in such a way that a material particle is "*affected*" only for what is inside of a small neighbourhood of the particle. There also exists a non-local treatment of continuum media iniciated by A. C. Eringen [40–43] based on the assumption that the stress on a material particle depends on all the body points of the continuum material.
Elastic simple bodies [93] are characterized under the assumption that the constitutive law depends on a point only on the infinitesimal deformation at the same point. Thus, a *mechanical response* for an elastic simple material $\mathcal{B}$ in a given reference configuration ϕ_0 is mathematically formalized as a differentiable map W from the set $\mathcal{B} \times Gl\left(3, \mathbb{R}\right)$, where $Gl\left(3, \mathbb{R}\right)$ is the general linear group of 3×3-regular matrices, to a fixed (finite dimensional) vector space V. We should clarify how the mechanical response changes with the changing of reference configuration. Let ϕ_1 be another configuration and W_1 be the mechanical response associated to ϕ_1. Then, we will impose that for any other (local) configuration ϕ

$$W\left(X, F_0\right) = W_1\left(X, F_1\right), \tag{2.4}$$

where F_i, $i = 0, 1$, is the associated matrix to the $1-$jet at $\phi_i\left(X\right)$ of $\phi \circ \phi_i^{-1}$. Hence, obviously Eq. (2.4) implies that

$$W_1\left(X, F\right) = W\left(X, F \cdot C_{01}\right), \tag{2.5}$$

for all regular matrix F where C_{01} is the associated matrix to the $1-$jet at $\phi_0\left(X\right)$ of $\phi_1 \circ \phi_0^{-1}$. So, Eq. (2.5) defines the *rule of change of reference configuration* of the mechanical response. Notice that, Eq. (2.5) permits us to define W as a map on the space of 1-jets of (local) configurations which is independent on the chosen reference configuration. In fact, for each configuration ϕ we could define

$$W\left(j^1_{X,x}\phi\right) = W\left(X, F\right),$$

where F is the associated matrix to the $1-$jet at $\phi_0\left(X\right)$ of $\phi \circ \phi_0^{-1}$.
It is remarkable that for any subbody $\mathcal{U}$ the mechanical response can be

restricted to $\mathcal{U}$. So, the structure on the body induces a structure of elastic simple body over each subbody.

From now on we will refer to $\mathcal{B}$ simply as a *body.*

The locality of the mechanical response implies that we may talk about the material at each point of the body. In this sense, given two body points a natural question arises: when are they made of the same material? Let us assume that body $\mathcal{B}$ may be undergone to experimetal observations under a fixed reference configuration ϕ_0 and body points X and Y are made of the same material. Looking at them under the microscope we do not need to see exactly the same. It could happen that the arrangement surrounding one point is changed with respect to the other.
In our mathematical framework this means that the constitutive equation of one of them differs from the other only by an application of a linear transportation. These kind of linear isomorphisms are called *material isomorphisms.*

Definition 2.1.3. Let $\mathcal{B}$ be a body. Two material particles $X, Y \in \mathcal{B}$ are said to be *materially isomorphic* if there exists a local diffeomorphism ψ from an open neighbourhood $\mathcal{U} \subseteq \mathcal{B}$ of X to an open neighbourhood $\mathcal{V} \subseteq \mathcal{B}$ of Y such that $\psi(X) = Y$ and

$$W(X, F \cdot P) = W(Y, F), \tag{2.6}$$

for all infinitesimal deformation F where P is given by the Jacobian matrix of $\phi_0 \circ \psi \circ \phi_0^{-1}$ at $\phi_0(X)$. The $1-$jets of local diffeomorphisms satisfying Eq. (2.6) are called *material isomorphisms.* A material isomorphism from X to itself is called a *material symmetry.* In cases where it causes no confusion we often refer to associated matrix P as the material isomorphism (or symmetry).

Notice that, the identities at the vector spaces $T_X\mathcal{B}$ are obviously material isomorphisms. On the other hand, for any material isomorphism P the inverse P^{-1} is again a material isomorphism. In fact, by using Eq. (2.6)

$$W(X, F) = W\left(X, F \cdot P^{-1} \cdot P\right) = W\left(Y, F \cdot P^{-1}\right).$$

Finally, the composition preserves material isomorphisms. So, the relation of being "*materially isomorphic*" defines an equivalence relation (symmetric, reflexive and transitive) over the body manifold $\mathcal{B}$.
For any body point X we denote by $G(X)$ the set of all material symmetries at X. Then, as a consequence we have that every $G(X)$ is a group. Therefore, it is trivial to prove that the material symmetry groups of materially isomorphic body points are conjugated, i.e., if X and Y are material isomorphic we have that

$$G(Y) = P \cdot G(X) \cdot P^{-1},$$

where P is a material isomorphism from X to Y.

Proposition 2.1.4. *Let $\mathcal{B}$ be a body. Two body points X and Y are materially isomorphic if, and only if, there exist two (local) configurations ϕ_1 and ϕ_2 such that*

$$W_1(X, F) = W_2(Y, F), \ \forall F,$$

where W_i is the mechanical response associated to ϕ_i for $i = 1, 2$.

Proof. Two body points X and Y are materially isomorphic if there exists a local diffeomorphism ψ from X to Y such that

$$W(X, F \cdot P) = W(Y, F), \tag{2.7}$$

for all infinitesimal deformation F where P is given by the induced tangent map of $\phi_0 \circ \psi \circ \phi_0^{-1}$ at $\phi_0(X)$. Then, we define the local configuration $\phi_1 = \phi_0 \circ \psi$. Then, by using Eq. (2.5) and Eq. (2.7), the local mechanical response W_1 induced by ϕ_1 satisfies that

$$W_1(X, F) = W(X, F \cdot P) = W(Y, F).$$

□

This result gives us an intuition behind the notion of material isomorphism. In fact, two point will be made of the same material if the mechanical response is the same under the action of two (possibly different) reference configurations. Furthermore, as a corollary we have the following immediate result: *Condition of being materially isomorphic is configuration indifferent.*

Definition 2.1.5. A body $\mathcal{B}$ is said to be *uniform* if all of its body points are materially isomorphic.

Roughly speaking, a body is uniform if there are not two different materials inside the body. Notice that the definition of uniformity is a *pointwise* property. In fact, consider a uniform body $\mathcal{B}$ and a fixed body point X_0, for any other body point Y we may choose a material isomorphism from Y to X_0, say $P(Y) \in Gl(3, \mathbb{R})$. So, we can construct a map $P : \mathcal{B} \to Gl(3, \mathbb{R})$ consisting of material isomorphisms. Nevertheless, P does not have to be differentiable. In other words, even when the body is uniform, the choice of the material isomorphisms is not, necessarily, smooth.

Definition 2.1.6. A body $\mathcal{B}$ is said to be *smoothly uniform* if for each point $X \in \mathcal{B}$ there is a neighbourhood $\mathcal{U}$ around X and a smooth map $P : \mathcal{U} \to Gl(3, \mathbb{R})$ such that for all $Y \in \mathcal{U}$ it satisfies that $P(Y)$ is a material isomorphism from Y to X. The map P is called a *left (local) smooth field of material isomorphisms.* A *right (local) smooth field of material isomorphisms* will be a smooth map $P : \mathcal{U} \to Gl(3, \mathbb{R})$ such that for all $Y \in \mathcal{U}$ it satisfies that $P(Y)$ is a material isomorphism from X to Y.

Note that a given map $P : \mathcal{U} \to Gl(3, \mathbb{R})$ is a left smooth field of material isomorphisms if, and only if, the map $P^{-1} : \mathcal{U} \to Gl(3, \mathbb{R})$, such that $P^{-1}(Y)$ is the inverse of $P(Y)$, is a right smooth field of material isomorphisms. Hence, smooth uniformity may be equivalently characterized by using right smooth fields of material isomorphisms.
Assume that P is a right (local) smooth field of material isomorphisms. Then, the mechanical response of the subbody $\mathcal{U}$ satisfies that

$$W(Y, F) = W(X, F \cdot P(Y)),$$

for all $Y \in \mathcal{U}$. Then, defining

$$\overline{W}(F) = W(X, F),$$

we have that

$$W(Y, F) = \overline{W}(F \cdot P(Y)). \tag{2.8}$$

The meaning of Eq. (2.8) is that the dependence of the mechanical response (near to a material particle) of the body coordinates is given

by a multiplication of F to the right by a right smooth field of material isomorphisms.
The following result shows that Eq. (2.8) defines a condition strictly weaker than the condition of being smoothly uniform.

Proposition 2.1.7. *Let $\mathcal{B}$ be a body. Then, suppose that the constitutive law W of $\mathcal{B}$ respect to a reference configuration ϕ_0 has associated a differentiable map $\overline{W} : Gl(3,\mathbb{R}) \to V$ satisfying Eq. (2.8) for a differentiable map $P : \mathcal{U} \to Gl(3,\mathbb{R})$. Then, P is a smooth field of material isomorphisms if, and only if,*

$$\overline{W}(F) = \overline{W}(F \cdot P(X)). \tag{2.9}$$

where X is a fixed point at the domain of P.

Proof. Assume that Eq. (2.9) is satisfied. Then, by Eq. (2.8)

$$\overline{W}(F \cdot P(X)) = W(X, F).$$

On the other hand, the same identity proves that for any $Y \in \mathcal{U}$

$$\overline{W}(F) = W\left(Y, F \cdot P(Y)^{-1}\right).$$

Hence, Eq. (2.9) implies that

$$W(X, F) = W\left(Y, F \cdot P(Y)^{-1}\right),$$

i.e., $P(Y)$ defines a material isomorphism from X to Y. The converse is trivial. □

Notice that condition Eq. (2.9) is not so strong. For instance, if the smooth fields P go through the identity matrix, then Eq. (2.9) is immediately fulfilled.
Let $\mathcal{B}$ be a smoothly uniform body and $P : \mathcal{U} \to Gl(3,\mathbb{R})$ be a left (local) smooth field of material isomorphisms. Then, we have a tool to compare vectors at $\mathcal{U}$. In this sense, Two tangent vectors V_{X_1} and V_{X_2} at two different material points X_1 and X_2 of $\mathcal{U}$ will be called *materially parallel*

with respect to P if they have the same components by the action of P. In other words,

$$\left[P\left(X_2\right)^{-1}\circ P\left(X_1\right)\right]\left(V_{X_1}\right)=V_{X_2}.$$

Here, for each $Y\in\mathfrak{U}$ and $V_Y\in T_Y\mathcal{B}$, $P\left(Y\right)\left(V_Y\right)$ is given by the action of P on the vector V_Y. In particular, considering ψ^Y as a local diffeomorphism from Y to X such that $P\left(Y\right)$ is the associated matrix of $j^1_{Y,X}\psi^Y$ under the composition of the reference configuration, it satisfies that

$$P\left(Y\right)\left(V_Y\right)=T_Y\psi^Y\left(V_Y\right).$$

A vector field $\Theta\in\mathfrak{X}\left(\mathfrak{U}\right)$ is *materially constant with respect to* P if for any two points X_1 and X_2 we have that $\Theta\left(X_1\right)$ and $\Theta\left(X_2\right)$ are materially parallel with respect to P. Equivalently, Θ is materially constant with respect to P if, and only if, the vector

$$\left[P\left(Y\right)\right]\left(\Theta\left(Y\right)\right),$$

does not depend on the choice of $Y\in\mathfrak{U}$.
Let us express this condition locally. Let $\left(X^I\right)$ be the local coordinates generated by the reference configuration ϕ_0. Then, for each A we define the local vector field P^A on $\mathcal{B}$ given by

$$P_A\left(Y\right)=P\left(Y\right)^{-1}\left(\frac{\partial}{\partial X^A_{|X}}\right).$$

Locally,

$$P_A=\left(P^{-1}\right)^B_A\frac{\partial}{\partial X^B},\tag{2.10}$$

where,

$$\left(P^{-1}\right)^B_A\left(Y\right)=\frac{\partial\left(X^B\circ\left(\psi^Y\right)^{-1}\right)}{\partial X^A_{|X}},$$

with $P\left(Y\right)=j^1_{Y,X}\psi^Y$. Then, Θ is materially constant with respect to P if, and only if, $\Theta^B P^A_B$ is constant for all A where P^A_B is the inverse matrix to $\left(P^{-1}\right)^B_A$, i.e.,

$$P^B_A\left(Y\right)=\frac{\partial\left(X^B\circ\psi^Y\right)}{\partial X^A_{|Y}}.$$

Thus, Θ is materially constant if it satisfies that

$$P_B^A \frac{\partial \Theta^B}{\partial X^I} + \Theta^B \frac{\partial P_B^A}{\partial X^I} = 0, \ \forall I$$

i.e.,

$$\frac{\partial \Theta^B}{\partial X^I} + \left(P^{-1}\right)_A^B \frac{\partial P_k^A}{\partial X^I} \Theta^k = 0, \ \forall B, I \tag{2.11}$$

We could even find another characterization by using the so-called *material connections*. For a brief explanation on connections as covariant derivatives see Box 2.1. The material connection associated to P is given by the unique covariant derivative ∇^P on $\mathcal{B}$ satisfying that

$$\nabla^P_{P_B} P_A = 0, \ \forall A, B.$$

A straightforward but tedious calculation shows us that th *Christoffel symbols of* ∇^P are given by

$$\Gamma^K_{IJ} = \left(P^{-1}\right)^K_L \frac{\partial P^L_I}{\partial X^J}. \tag{2.12}$$

Therefore, Θ is materially constant with respect to P if, and only if, $\nabla^P \Theta = 0$. It is important to note that the material connections are configuration-indifferent (see [54] or section 3.1 for a proof).
Material connections have been masterfully treated by Wang in [92]. In [93] various examples of material connections with non-vanishing curvature are studied (Chapter 5). Other ways to construct material connections will be presented in section 3.1 [54].

Definition 2.1.8. Let M be a manifold. A *derivation on* M is a $\mathbb{R}-$linear map $D : \mathfrak{X}(M) \to \mathfrak{X}(M)$ with a vector field $\Theta \in \mathfrak{X}(M)$ such that for each $f \in \mathcal{C}^{\infty}(M)$ and $\Xi \in \mathfrak{X}(M)$,

$$D(f\Xi) = fD(\Xi) + \Theta(f)\Xi.$$

We call Θ the *base vector field of* D. So, a derivation on M is characterized by two geometrical objects, D and Θ.

A classical example of derivation is given by the bracket of vector fields on a manifold M. In fact, let Θ be a vector field on M, the operator given by fixing Θ in the Lie bracket

$$[\Theta, \cdot] : \mathfrak{X}(M) \to \mathfrak{X}(M),$$

is a derivation on M with Θ as base vector field.
Another example comes from the so called *covariant derivatives*. A covariant derivative on M is a $\mathbb{R}-$bilinear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that,

(1) It is $\mathcal{C}^{\infty}(M)-$linear in the first variable.
(2) For all $\Theta, \Xi \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^{\infty}(M)$,

$$\nabla_{\Theta} f\Xi = f\nabla_{\Theta}\Xi + \Theta(f)\Xi. \tag{2.13}$$

Then, any vector field $\Theta \in \mathfrak{X}(M)$ generates a derivation on A, ∇_{Θ}, (with base vector field Θ) fixing the first coordinate again, i.e.,

$$\nabla_{\Theta} : \mathfrak{X}(M) \to \mathfrak{X}(M),$$

such that

$$\nabla_{\Theta}(\Xi) = \nabla_{\Theta}\Xi, \ \forall \Xi \in \mathfrak{X}(M).$$

Associated to any covariant derivative ∇ there are two important tensors:

- **Torsion:** $T(\Theta, \Xi) = \nabla_{\Theta}\Xi - \nabla_{\Xi}\Theta - [\Theta, \Xi], \ \forall \Theta, \Xi \in \mathfrak{X}(M).$
- **Curvature:** $R(\Theta, \Xi)\chi = \nabla_{\Theta}\nabla_{\Xi}\chi - \nabla_{\Xi}\nabla_{\Theta}\chi - \nabla_{[\Theta,\Xi]}\chi, \ \forall \Theta, \Xi, \chi \in \mathfrak{X}(M).$

A covariant derivative is said to be *flat* if its curvature is zero.

Lemma 2.1.9. *Let ∇ be a covariant derivative on a manifold M. ∇ is flat and torsion-free if, and only if, there exists an atlas $\left(x^i\right)$ of M such that*

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0.$$

In general, for each local coordinates (x^i) on M,

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

The local functions Γ^k_{ij} are called *Christoffel symbols of* ∇ *respect to* (x^i).

Box 2.1: Derivations and Covariant Derivations

We have defined the uniformity as the mathematical formalization of the following statement: *All the points are made of the same material.* Smooth uniformity consisted in a light imposition of smoothness on the body. A new more restrictive property is presented as the *local homogeneity of the body.*
In Noll's terminology, the notion of (local) homogeneity coincides with the absence of defects of the body. So, in a purely physical point of view, it is conceivable that the absence of defects may be fulfilled for non-uniform bodies. Homogeneity for non-uniform bodies has been properly defined in [39, 53] (see section 4.2).
The classical definition of (local) homogeneity means that the body may be depicted in such a configuration in which the translations of any point to any other are material isomorphisms. In other words, there exists a configuration which satisfies that all the points are indistinguishable from each other as far as the mechanical response concerned.

Definition 2.1.10. A body $\mathcal{B}$ is said to be *homogeneous* if it admits a global configuration ϕ which induces a left global smooth field of material isomorphisms P such that $P(Y)$ is the associated matrix to the $1-$jet

$$j^1_{Y,X}\left(\phi^{-1} \circ \tau_{\phi(X)-\phi(Y)} \circ \phi\right), \tag{2.14}$$

via the reference configuration ϕ_0, for all body point $Y \in \mathcal{B}$ and a fixed $X \in \mathcal{B}$ where $\tau_{\phi(X)-\phi(Y)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\phi(X) - \phi(Y)$. $\mathcal{B}$ is said to be *locally homogeneous* if there exists a covering of $\mathcal{B}$ by homogeneous open sets. $\mathcal{B}$ is said to be *(locally) inhomogeneous* if it is not (locally) homogeneous.

A left (local) smooth field of material isomorphisms P is said to be *integrable* if it is induced by a (local) configuration ϕ via Eq. (2.14). These kind of configuration are called *homogeneous configurations.*
As a corollary of proposition 2.1.7 we have the following result.

Proposition 2.1.11. *Let $\mathcal{B}$ be a body. Then, $\mathcal{B}$ is locally homogeneous if, and only if, there exist (local) reference configurations such that for the associated constitutive laws W there are differentiable maps $\overline{W}$: $Gl(3,\mathbb{R}) \to V$ satisfying that*

$$W(Y, F) = \overline{W}(F),$$

for all body point Y in the domain and $F \in Gl(3,\mathbb{R})$.

Therefore, a material body is homogeneous if there exists a configuration such that the material response does not depend on the body points.

Let $\mathcal{B}$ be a (local) homogeneous body. Then, considering a homogeneous configuration as the reference configuration, the coordinates $\left(P^{-1}\right)_A^B$ of the (local) vector fields P_A given in Eq. (2.10) satifying that

$$\left(P^{-1}\right)_A^B = \delta_A^B.$$

Then, the Christoffel symbols of the material conection ∇^P are zero respect to the homogeneous configuration. So, P is integrable if there exists a local system of coordinates on $\mathcal{B}$ such that the partial derivatives are materially constant vector fields with respect to P.
Therefore, the material connection of P provides a way to evaluate whether the smooth field of material isomorphisms P is integrable: P is integrable if, and only if, ∇^P is a flat and torsion-free covariant derivative (see lemma 2.1.9).
There is still another treatment of homogeneity by using the theory of $G-$structures in which the (local) homogeneity corresponds to the integrability of a particular $G-$structure. This approach can be found in [31] (see [32] or [92]; see also [6] and [69]).
Let $\mathcal{B}$ be a smoothly uniform body. Fix $Z_0 \in \mathcal{B}$ and $\overline{Z}_0 = j^1_{0,Z_0}\phi \in F\mathcal{B}$ a frame at Z_0. Consider the following set:

$$\omega_{G_0}(\mathcal{B}) := \{j^1_{Z_0,Y}\psi \cdot \overline{Z}_0, \ : \ j^1_{Z_0,Y}\psi \text{ is a material isomorphism}\}.$$

Then, $\omega_{G_0}(\mathcal{B})$ is a G_0-structure on $\mathcal{B}$ (which contains $\overline{Z}_0$). In fact (see proposition 2.3.48),

$$\omega_{G_0}(\mathcal{B}) = \Omega_{Z_0}(\mathcal{B}) \cdot \overline{Z}_0.$$

Notice that the structure group of $\omega_{G_0}(\mathcal{B})$ is given by,

$$G_0 := \overline{Z}_0^{-1} \cdot G(Z_0) \cdot \overline{Z}_0.$$

A local section of $\omega_{G_0}(\mathcal{B})$ will be called *local uniform reference*. A global section of $\omega_{G_0}(\mathcal{B})$ will be called *global uniform reference*. We call *reference crystal* to any frame $\overline{Z}_0 \in F\mathcal{B}$ at Z_0.

Remark 2.1.12. (1) If we change the point Z_0 to another body point Z_1 then we obtain an isomorphic G_0-structure. We only have to take a frame $\overline{Z}_1$ as the composition of $\overline{Z}_0$ with a material isomorphism $j^1_{Z_0,Z_1}\psi$.

(2) We have fixed a configuration ϕ_0. Suppose that ϕ_1 is another reference configuration such that the change of configuration is given by $\psi = \phi_1^{-1} \circ \phi_0$. Transporting the reference crystal $\overline{Z}_0$ via $F\psi$ we get another reference crystal such that the G_0-structures are isomorphic.

(3) Finally suppose that we have another crystal reference $\overline{Z}'_0$ at Z_0. Hence, the new G'_0-structure, $\omega_{G'_0}(\mathcal{B})$, is conjugate of $\omega_{G_0}(\mathcal{B})$, namely,

$$G'_0 = A \cdot G_0 \cdot A^{-1}, \quad \omega_{G'_0}(\mathcal{B}) = \omega_{G_0}(\mathcal{B}) \cdot A,$$

with $A = \overline{Z}'_0 \cdot \overline{Z}_0^{-1}$.

In this way, the definition of homogeneity in terms of $G-$structures is the following,

Definition 2.1.13. A body $\mathcal{B}$ is said to be *homogeneous* with respect to a given frame $\overline{Z}_0$ if it admits a global deformation κ such that κ^{-1} induces a uniform reference P, i.e., for each $X \in \mathcal{B}$

$$P(X) = j^1_{0,X}\left(\kappa^{-1} \circ \tau_{\kappa(X)}\right),$$

where $\tau_{\kappa(X)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\kappa(X)$. $\mathcal{B}$ is said to be *locally homogeneous* if every $X \in \mathcal{B}$ has a neighbourhood which is homogeneous.

This definition is equivalent to the first one. The proof is included in section 3.1 in term of Lie groupoids [54].
It is easy to prove the following result:

Proposition 2.1.14. *If $\mathcal{B}$ is homogeneous then $\omega_{G_0}(\mathcal{B})$ is integrable. Conversely, $\omega_{G_0}(\mathcal{B})$ is integrable implies that $\mathcal{B}$ is locally homogeneous.*

Notice that, using remark 2.1.14, this result shows us that the homogeneity does not depend on the point, the reference configuration and the frame $\overline{Z}_0$. It will not happen the same with *Cosserat materials* which we will introduce later on (see proposition 2.1.24).
Let us now present some examples of simple elastic bodies. More examples will be estudied along the memory.

Solids: Let $\mathcal{B}$ be a body with reference configuration ϕ_0. A material particle X is said to be *solid* if it satisfies that the group

$$G_0^X = j_{X,0}^1 \left(\tau_{-\phi_0(X)} \circ \phi_0\right) \cdot G(X) \cdot j_{0,X}^1 \left(\phi_0^{-1} \circ \tau_{\phi_0(X)}\right),$$

called *material symmetry group of X respect to ϕ_0* is a conjugated subgroup of a subgroup of the orthogonal group $\mathcal{O}$ where $j_{0,X}^1\left(\phi_0^{-1} \circ \tau_{\phi_0(X)}\right)$ is the linear frame on $\mathcal{B}$ induced by $j_{0,X}^1\left(\phi_0^{-1} \circ \tau_{\phi_0(X)}\right)$ (see appendix A). $\mathcal{B}$ is said to be *solid* if all its body points are solids.
It is usual to assume that there exist (local) configurations ϕ_1 such that the symmetry groups respect to ϕ_1 are sugroups of $\mathcal{O}$ (*contorted aelotropy*). Notice that, punctually, this is always true. These kind of configurations are called *undistorted states*.
Thus, let $\mathcal{B}$ be a solid with a reference configuration ϕ_0 which is an undistorted state. The pullback of the usual metric on $\mathbb{R}^3$ by ϕ_0 results into a Riemannian metric g_0 on $\mathcal{B}$.
Let $j_{X,X}^1\psi$ be a material symmetry. Then, the associated matrix

to $j^1_{0,0}\left(\tau_{-\phi_0(X)} \circ \phi_0 \circ \psi \circ \phi_0^{-1} \circ \tau_{\phi_0(X)}\right)$ is an orthogonal matrix. In particular,

$$T_{\phi_0(X)}\left(\phi_0 \circ \psi \circ \phi_0^{-1}\right)(v) \cdot T_{\phi_0(X)}\left(\phi_0 \circ \psi \circ \phi_0^{-1}\right)(w) = v \cdot w,$$

for all $v, w \in \mathbb{R}^3 \cong T_{\phi_0(X)}\mathbb{R}^3$ where $\cdot$ is the scalar product in $\mathbb{R}^3$. Hence, by definition, for any two vector $V_X, W_X \in T_X\mathcal{B}$ we have that

$$\begin{aligned} & g_0(X)\left(T_X\psi(V_X), T_X\psi(W_X)\right) = \\ = \; & T_X(\phi_0 \circ \psi)(V_X) \cdot T_X(\phi_0 \circ \psi)(W_X) \\ = \; & T_X\phi_0(V_X) \cdot T_X\phi_0(W_X) \\ = \; & g_0(X)(V_X, W_X) \end{aligned}$$

In other words, the materials symmetries are isometries for the metric g.
Now, suppose that $\mathcal{B}$ *isotropic*, i.e., G_0^X is a conjugated group of $\mathcal{O}$ for all body point X. In this case, the same reasoning proves that a local automorphism ψ induces a material symmetry at a point X if, and only if, ψ induces an isometry at X of g_0. So, there exist Riemannian metrics characterizing the material symmetries.
If the solid $\mathcal{B}$ is supposed to be smoothly uniform, we do not need to assume the existence of global (or local) undistorted states to constructs these kind of metrics.
Consider a left (local) smooth field of material isomorphisms P around a material particle X and a local configuration ϕ such that it is an undistorted state at X, i.e.,

$$G = j^1_{X,0}\left(\tau_{-\phi(X)} \circ \phi\right) \cdot G(X) \cdot j^1_{0,X}\left(\phi^{-1} \circ \tau_{\phi(X)}\right),$$

is a subgroup of $\mathcal{O}$. Notice that, punctually, the undistorted states always exist.
Then, we define a (local) riemannian metric g^P on $\mathcal{B}$ as follows,

$$g^P(Y)(V_Y, W_Y) = T_X\phi\left[P(Y)(V_Y)\right] \cdot T_X\phi\left[P(Y)(W_Y)\right] \tag{2.15}$$

We may assume (composing by the left with $P(X)^{-1}$) that $P(X)$ is the identity at $T_X\mathcal{B}$. Then, its satisfies that

$$g^P(Y)(V_Y, W_Y) = g^P(X)(P(Y)(V_Y), P(Y)(W_Y)). \tag{2.16}$$

Otherwise speaking, the values of the metric g^P is the combination of the values of the metric at a fixed point X and the translations by P. Notice that, Eq. (2.16) shows us that the composition $T_X\phi \circ P$ is a *left (local) field of undistorted states*. In fact, for any material isomorphism $j^1_{Y,Z}\psi$ we have that the composition $P(Z) \circ T_Y\psi \circ P^{-1}(Y)$ defines an isometry for g^P. Here $P(Z)$ is being considered as the 1−jet $j^1_{Z,X}\psi^Z$ such that the associated matrix via the composition with the reference configuration is $P(Z)$.
So, the smooth uniformity permits us to extend differentiably any undistorted state at a fixed point to a field of undistorted states.
In this case we have that the material isomorphisms are isometries of g^P.
Suppose that the Levy-Civita connection associated to g^P is flat and torsion-free. Equivalently, there exists a system of coordinates (y^i) on $\mathcal{B}$ such that $g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \delta^i_j$ (see for instance [78]). Then, by using Eq. (2.15) there exists a local chart $\varphi = (y^j)$ such that $T_X\phi \circ P(Y) \circ T_y\varphi^{-1}$ induces a orthogonal matrix for all $Y = \varphi^{-1}(y)$ in the domain. This fact implies ($P(X)$ is the identity) that

$$T_X\phi \circ T_x\varphi^{-1},$$

induces also a orthogonal matrix with $X = \varphi^{-1}(x)$. Therefore, φ is indeed a local undistorted state. Thus, we have proved that the existence of local undistorted states is equivalent to that the Levi-Civita connection pf g^P is flat and torsion-free.
Assume now that the material body $\mathcal{B}$ is furthermore isotropic. Then, all the isometries are material isomorphisms and, hence, φ induces a (local) smooth field of material isomorphisms via Eq. (2.14). Accordingly, $\mathcal{B}$ *is locally homogeneous if, and only if, the Levi-Civita connection of* g^P *is flat and torsion-free.*

Fluids: Let $\mathcal{B}$ be a body with reference configuration ϕ_0. A material particle X is said to be *elastic fluid* if it satisfies that the material symmetry group of X respect to ϕ_0 is the unimodular group, i.e., the group of matrices with unit determinant. Notice that it does not make sense to take a conjugation of the unimodular group because any conjungation (by any non-singular matrix) of the unimodular group is the unimodular group. $\mathcal{B}$ is said to be *elastic fluid* if all its body points are elastic fluids.

Let g_0 be the riemannian metric on $\mathcal{B}$ defined above and V_0 be its associated volume form. Then, it is easy to check that $j^1_{X,X}\psi$ is a material symmetry if, and only if, ψ preserves V_0 at X, i.e.,

$$\psi^* V_0 (X) = V_0 (X).$$

Now, let us assume that $\mathcal{B}$ is smoothly uniform. Let P be a left (local) smooth field of material isomorphisms P around a material particle X. So, we may consider the metric g^P defined above. Then, analogously to solids, we may prove that $j^1_{Y,Z}\psi$ is a material isomorphism if, and only if, ψ preserves the volume form V^P of g^P at Y, i.e.,

$$\psi^* V^P (Y) = V^P (Z).$$

Therefore, immediately we have that any configuration induces a smooth field of material isomorphisms via Eq. (2.14). In other words, *any smoothly uniform elastic fluid is homogenous.*

Cosserat medium

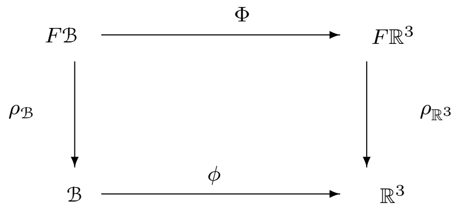
We have already established the necessary fundamental notions for simple material bodies. Nevertheless, several non-simple material can be found. Indeed, rocks or bones cannot be properly modeled without taking into account extra kinematical variables [10]. Eugène and François Cosserat presented a theory of the so-called *generalized media* between 1905 and 1910. Here, to each point Cosserats associated a family of vectors called *directors*. Mathematically speaking, a *simple elastic Cosserat continuum*

or, simply, *Cosserat continuum* is given by a manifold $\mathcal{B}$ and a family of independent vector fields on $\mathcal{B}$. Important developments may be found in Maugin [68, 70] or in the Proceedings [60] edited by Kröner; see also Eringen [43].
The necessary geometrical structures to develop a rigurous theory is closely related with the notion of frame bundle. Thus, we can intepret a Cosserat medium as a linear frame bundle $F\mathcal{B}$ of a manifold $\mathcal{B}$ which can be covered with just one chart (see [37]). $\mathcal{B}$ is usually called the *macromedium* or *underlying body*. With some abuse of notation, we shall call $\mathcal{B}$ the *Cosserat continuum*.
The points in $F\mathcal{B}$ will be denoted by $\overline{X}, \overline{Y}, \overline{Z} \in F\mathcal{B}$. However, we will call *body point* to the points on the underlying body $\mathcal{B}$. The points in $F\mathbb{R}^3$ will be denoted by lower case letters ($\overline{x}, \overline{y}, \overline{z} \in F\mathbb{R}^3$). The *spatial points* will be the points at $\mathbb{R}^3$.
A *Cosserat subbody* of $F\mathcal{B}$ is a open subset $\overline{\mathcal{U}}$ of the Cosserat medium $F\mathcal{B}$. We will usually consider subbodies given by $F\mathcal{U}$ with $\mathcal{U}$ a open subset of $\mathcal{B}$.
Let $\rho_{\mathcal{B}} : F\mathcal{B} \to \mathcal{B}$ be the canonical projection of the frame bundle $F\mathcal{B}$ over $\mathcal{B}$. Then, for any body point $X \in \mathcal{B}$, the fibre $\rho_{\mathcal{B}}^{-1}(X)$ is given by all the possible basis of $T_X\mathcal{B}$. From the physical point of view, each fibre contains the information about what happens at a "*grain*" level. As the model conceived by the Cosserat brothers, that information should be localized into any particular basis. This fact will be taken into consideration in the definitions of *configurations* and *deformations*.
To face the problem of finding an appropriate definition of configuration we will have to take into account all the geometric structure in the Cosserat medium $F\mathcal{B}$. We want to start defining a configuration as a map

$$\Phi : F\mathcal{B} \to F\mathbb{R}^3.$$

However, any arbitrary embedding of this type will not work. So, we will need to impose some other conditions:

(i) Compatibility with the macromedium: The configuration Φ should have incorporated a configuration ϕ of the underlying body (i.e., an embedding $\phi : \mathcal{B} \to \mathbb{R}^3$) such that the fibres do not mix up. Particularly, K turns a fibre at a body point X into a fibre at $\phi(X)$. Physically, we want that any "*grain*" carries its own information into its image in the configuration. This is mathematically expressed as the commutativity of the following diagram



i.e.,

$$\rho_{\mathbb{R}^3} \circ \Phi = \phi \circ \rho_{\mathcal{B}}.$$

(i) Configuration-independence: Any particular representation of a grain in the physical space should be an intrinsic quantity, i.e., it should not depend on a triple which can be chosen to depict the grain. Mathematically speaking, we will impose that the natural (right) action on the frame bundle commutes with the configuration Φ.

Putting together all these properties we obtain the following definition.

Definition 2.1.15. Let $F\mathcal{B}$ be a Cosserat medium. A *configuration* of $F\mathcal{B}$ is an embedding $\Phi : F\mathcal{B} \to F\mathbb{R}^3$ of principal bundles such that the induced Lie group morphism $\tilde{\phi} : Gl\left(3, \mathbb{R}\right) \to Gl\left(3, \mathbb{R}\right)$ is the identity map.

This means that Φ satisfies

$$\Phi\left(\overline{X} \cdot g\right) = \Phi\left(\overline{X}\right) \cdot g, \ \forall \overline{X} \in F\mathcal{B}, \ \forall g \in Gl\left(3, \mathbb{R}\right).$$

Also, as we wanted, Φ induces an embedding $\phi : \mathcal{B} \to \mathbb{R}^3$ verifying

$$\rho_{\mathbb{R}^3} \circ \Phi = \phi \circ \rho_{\mathcal{B}},$$

with $\rho_{\mathbb{R}^3} : F\mathbb{R}^3 \to \mathbb{R}^3$ (resp. $\rho_{\mathcal{B}} : F\mathcal{B} \to \mathcal{B}$) denotes the canonical projection of the frame bundle $F\mathbb{R}^3$ (resp. $F\mathcal{B}$). In particular, ϕ is a configuration of the macromedium $\mathcal{B}$.
Notice that the subbundle $\Phi(F\mathcal{B})$ of $F\mathbb{R}^3$ is just the frame bundle of $\phi(\mathcal{B})$, i.e.,

$$\Phi(F\mathcal{B}) = F\phi(\mathcal{B}).$$

Since we are dealing with equivariants embedding, we can consider equivalence classes of the $1-$jets $j^1_{\overline{X},\Phi(\overline{X})}\Phi$ according to the action (2.28). So, the equivalence class of an $1-$jet $j^1_{\overline{X},\Phi(\overline{X})}\Phi$, which is denoted by $j^1_{X,\phi(X)}\Phi$ like in the non-holonomic groupoid of second order, is called *infinitesimal configuration at* X. Analogously to the case of simple bodies, a configuration Φ_0 is usually fixed. This configuration is called *reference configuration.*
Let $\phi : \mathcal{B} \to \mathbb{R}^3$ be an embedding. Then, the first prolongation $F\phi$ of ϕ (see Eq. A.1 in appendix A) defines an embedding of principal bundles such that the induced Lie group morphism is the identity map, i.e., a configuration for the Cosserat body $F\mathcal{B}$. A *second-grade body* is also modelized on the manifold $F\mathcal{B}$ but there are less configurations. In fact, a configuration of the second-grade material $F\mathcal{B}$ is a first prolongations of an embedding from $\mathcal{B}$ to $\mathbb{R}^3$.
Note that there exists a canonical principal bundle isomorphism $l : F\mathbb{R}^n \to \mathbb{R}^n \times Gl(n, \mathbb{R})$ over the identity map such that

$$l\left(j^1_{0,x}\phi\right) = \left((x, J\phi_{|0})\right), \ \forall j^1_{0,x}\phi \in F\mathbb{R}^n,$$

where $J\phi_{|0}$ is the Jacobian matrix of ϕ at 0 (see appendix A). So, any configuration can be seen as a chart of $F\mathcal{B}$. In this way, coordinates in the reference configuration will be denoted by $\overline{X}^I$, while any other coordinates will be denoted by $\overline{x}^i$.
In terms of coordinates, a configuration of the Cosserat body $F\mathcal{B}$ is characterized by the following smooth maps,

$$x^i = \phi^i\left(X^J\right), \qquad \Phi^i_I = \Phi^i_I\left(X^J\right).$$

Thus, there are two indepenpent ways of taking vectors to $\mathbb{R}^3$ by means of a configuration Φ. Let $V_X \in T_X\mathcal{B}$ be a vector on a body point X. The first

way is via the configuration of the macromedium ϕ; $T_X\phi\left(V_X\right)$. The second way is associated to the "*micromedium*" or "*grain*" and is characterized by the action of the entries Φ_I^i over the coordinates of V_X.
In second-grade bodies these two mechanisms are exactly the same.

Definition 2.1.16. A *deformation* is a change of configurations, namely $\overline{\kappa} = \Phi \circ \Phi_0^{-1}$.

For a configuration a deformation $\overline{\kappa} = \Phi \circ \Phi_0^{-1}$, its class of 1−jets $j^1_{\phi_0(X),\phi(X)}\tilde{\kappa}$ is called an *infinitesimal deformation* at $\phi_0\left(X\right)$. Notice that the induced map of $\tilde{\kappa}$, $\kappa = \phi \circ \phi_0^{-1}$, is a deformation on the body $\mathcal{B}$.
Again, the material response will be assumed to be a *local property*, i.e., a body point is only "*affected*" for what happens inside of a small neighbourhood of the point. However, in the case of Cosserat media we should take into account the adding structure of frame bundle. Indeed, we will impose that the constitutive law "*feels*" both the macro and micro mechanics of dragging vectors, as well as the gradient of the last one. *Second grade materials* are recovered when the deformation of the grains is identified with the deformation gradient of the matrix.
Thus, the constitutive elastic law is now written as

$$W = W\left(X, \overline{F}\right),$$

where X is a material particle of the macromedium and $\overline{F}$ is the gradient of a deformation $\overline{\kappa}$ at a frame $\overline{X}$ on the point X. In term of local coordinates, by following notation introduced above, the entries of the matrix $\overline{F}$ are denoted as follows,

$$\overline{F}^i_J = \frac{\partial \overline{x}^i}{\partial \overline{X}^J} \tag{2.17}$$

Since $\overline{\kappa}$ is a morphism of principal bundles, $\overline{F}$ depends only on the base points.
We again should clarify how the mechanical response change with the changing of reference configuration. Let Φ_1 be another configuration and W_1 be the mechanical response associated to Φ_1. Then, we will impose that for any other (local) configuration Φ

$$W\left(X, \overline{F}_0\right) = W_1\left(X, \overline{F}_1\right), \tag{2.18}$$

where $\overline{F}_i$ is the associated matrix to the 1−jet at $\phi_i(X)$ of $\Phi \circ \Phi_i^{-1}$. Hence, obviously Eq. (2.18) implies that

$$W_1\left(X, \overline{F}\right) = W\left(X, \overline{F} \cdot \overline{C}_{01}\right), \tag{2.19}$$

for all regular matrix $\overline{F}$ where $\overline{C}_{01}$ is the associated matrix to the class 1−jet at $\phi_0(X)$ of $\Phi_1 \circ \Phi_0^{-1}$. So, Eq. (2.19) defines the rule of change of reference configuration of the mechanical response of the Cosserat medium $F\mathcal{B}$. Notice that, Eq. (2.19) permits us to define W on the space of class of 1-jets of (local) configurations. In fact, for each configuration Φ we could define

$$W\left(j^1_{X,x}\Phi\right) = W\left(X, \overline{F}\right),$$

where $\overline{F}$ is the associated matrix to the class of 1−jets at $\phi_0(X)$ of $\Phi \circ \Phi_0^{-1}$. For any Cosserat subbody $\overline{\mathcal{U}}$ the mechanical response may be restricted to $\overline{\mathcal{U}}$ endowing it with a structure of Cosserat medium.
One more time, due to the locality of the mechanical response, it is easy to realize that we may compare the material properties from one body point to another. In this sense, two points will be "*made of the same material*" if the material response is the same via a (local) transportation. These "*transportations*" again are called *material isomorphisms.*

Definition 2.1.17. Let $F\mathcal{B}$ be a Cossera medium. Two material particles $X, Y \in \mathcal{B}$ are said to be *materially isomorphic* if there exists a local principal bundle isomorphism over the identity map on $Gl(3,\mathbb{R})$, Ψ, from $F\mathcal{U} \subseteq F\mathcal{B}$ with $X \in \mathcal{U}$ to $F\mathcal{V} \subseteq F\mathcal{B}$ with $Y \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are open neighbourhood of $\mathcal{B}$, such that $\psi(X) = Y$ and

$$W\left(X, \overline{F} \cdot \overline{P}\right) = W\left(Y, \overline{F}\right), \tag{2.20}$$

for all infinitesimal deformation $\overline{F}$ where $\overline{P}$ is given by the associated matrix to the class of 1−jets of $\Phi_0 \circ \Psi \circ \Phi_0^{-1}$ at $\phi_0(X)$. The class of 1−jets of principal bundle isomorphisms satisfying Eq. (2.20) are called *material isomorphisms.* A material isomorphism from X to itself is called a *material symmetry.* If there is no room for confusion, we will refer to associated matrix $\overline{P}$ as the material isomorphism (or symmetry).

For any body point X we denote by $\overline{G}(X)$ the set of all material symmetries at X. Then, as a consequence of what we have proved, all $\overline{G}(X)$ are groups. In fact, it is obvious that the symmetry groups of materially isomorphic body points are conjugated, i.e., if X and Y are material isomorphic we have that

$$\overline{G}(Y) = \overline{P} \cdot \overline{G}(X) \cdot \overline{P}^{-1},$$

where $\overline{P}$ is the material isomorphism from X to Y. Analogously to the case of simple material, we prove the following result:

Proposition 2.1.18. *Let $F\mathcal{B}$ be a Cosserat medium. Two body points X and Y are materially isomorphic if, and only if, there exist two (local) configurations Φ_1 and Φ_2 such that*

$$W_1\left(X, \overline{F}\right) = W_2\left(Y, \overline{F}\right), \ \forall \overline{F},$$

where W_i is the mechanical response associated to Φ_i for $i = 1, 2$.

So, we have that the condition of being materially isomorphic is again configuration indifferent.

Definition 2.1.19. A Cosserat medium $F\mathcal{B}$ is said to be *uniform* if all of its body points are materially isomorphic.

Thus, uniformity means that the Cosserat continuum is made of one unique material. Adding conditions of "*smoothness*" we recover the smooth uniformity. So, a *left field of material isomorphisms around a body point* X (resp. *right field of material isomorphisms around a body point* X) is a map $\overline{P} : \mathcal{B} \to Gl\left(n + n^2, \mathbb{R}\right)$ such that for each $Y \in \mathcal{B}$, $P(Y)$ is material isomorphism from Y to X (resp. from X to Y).

Definition 2.1.20. A Cosserat $F\mathcal{B}$ is said to be *smoothly uniform* if for each point $X \in \mathcal{B}$ there is a neighbourhood $\mathcal{U}$ around X and a left smooth field of material isomorphism $\overline{P}$ around a body point X defined on $\mathcal{U}$.

Once again, the existence of left smooth field of material isomorphisms is equivalent to the existence of right smooth field of material isomorphisms via the inversion of matrices.
Observe that, by following an analogous process to simple media, we can easily prove the following result:

Proposition 2.1.21. *Let $F\mathcal{B}$ be a Cosserat medium with constitutive law W. Then, $F\mathcal{B}$ is smoothly uniform if, and only if, there exists a covering of $\mathcal{B}$ and for each of open set $\mathcal{U}$ of the covering there is a differentiable map $\overline{W} : Gl(12,\mathbb{R}) \to V$ satisfying that for all $Y \in \mathcal{U}$ and $\overline{F} \in Gl(12,\mathbb{R})$*

(i)

$$W\left(Y,\overline{F}\right) = \overline{W}\left(\overline{F}\cdot\overline{P}(Y)\right) \tag{2.21}$$

(ii)

$$\overline{W}\left(\overline{F}\right) = \overline{W}\left(\overline{F}\cdot\overline{P}(X)\right) \tag{2.22}$$

for some differentiable map $\overline{P} : \mathcal{U} \to Gl(12,\mathbb{R})$ and a fixed $X \in \mathcal{U}$.

The *homogeneity* in the case of Cosserat materials is rather more complicated. We will introduce the definition of homogeneity used in [37] where the authors discuss second-order non-holonomic $\overline{G}-$structures (see appendix A).

Assume that $F\mathcal{B}$ is smoothly uniform. Let be $\overline{Z}_0^2 = j^1_{e_1,\overline{Z}_0}\Phi \in \overline{F}^2\mathcal{B}$ a non-holonomic frame of second order at the body point Z_0. Define the set $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ given by the $1-$jets $j^1_{Z_0,Y}\Psi\cdot\overline{Z}_0^2$ such that $j^1_{Z_0,Y}\Psi$ is a material isomorphism.

Then, $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ is a non-holonomic $\overline{G}_0-$structure of second order on $\mathcal{B}$ (which contains $\overline{Z}_0^2$). In fact (see section 3.2),

$$\overline{\omega}_{\overline{G}_0}(\mathcal{B}) = \overline{\Omega}_{Z_0}(\mathcal{B})\cdot\overline{Z}_0^2.$$

Notice that the structure group of $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ is given by,

$$G_0 := \overline{Z}_0^{2^{-1}}\cdot\overline{G}(Z_0)\cdot\overline{Z}_0^2.$$

A local section of $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ will be called *local uniform reference.* A global section of $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ will be called *global uniform reference.* We call *reference crystal* to any frame $\overline{Z}_0^2 \in \overline{F}^2\mathcal{B}$ at Z_0. Observe that the canonical projection of the second-order non-holonomic $\overline{G}_0-$structure $\overline{\omega}_{\overline{G}_0}(\mathcal{B})$ is a G_0-structure denoted by $\omega_{G_0}(\mathcal{B})$.

Remark 2.1.22. (1) If we change the point Z_0 to another point Z_1 then we can obtain the same second-order non-holonomic $\overline{G}_0-$structure. We only have to take a frame $\overline{Z}_1^2$ as the composition of $\overline{Z}_0^2$ with a $j^1_{Z_0,Z_1}\Psi \in \overline{G}\left(Z_0, Z_1\right)$.

(2) We have fixed a configuration Φ_0. Suppose that Φ_1 is another reference configuration such that the change of configuration (or deformation) is given by $\tilde{\kappa} = \Phi_1^{-1} \circ \Phi_0$. Transporting the reference crystal $\overline{Z}_0^2$ via $F\tilde{\kappa}$ we get another reference crystal such that the second-order non-holonomic $\overline{G}_0-$structures are isomorphic.

(3) Finally suppose that we have another crystal reference $\overline{Z}_1^2$ at Z_0. Hence, the new second-order non-holonomic $\overline{G}_0'-$structure, $\overline{\omega}_{\overline{G}_0'}\left(\mathcal{B}\right)$, is conjugated of $\overline{\omega}_{\overline{G}_0}\left(\mathcal{B}\right)$, namely,

$$\overline{G}_0' = \overline{m} \cdot \overline{G}_0 \cdot \overline{m}^{-1}, \quad \overline{\omega}_{\overline{G}_0'}\left(\mathcal{B}\right) = \overline{\omega}_{\overline{G}_0}\left(\mathcal{B}\right) \cdot \overline{m},$$

for $\overline{m} = \overline{Z}_0^{2^{-1}} \cdot \overline{Z}_1^2$.

In this way, the definition of homogeneity is the following:

Definition 2.1.23. A Cosserat continuum $\mathcal{B}$ is said to be *homogeneous* with respect to the crystal reference $\overline{Z}_0^2$ if it admits a global deformation $\overline{\kappa}$ such that $\overline{\kappa}^{-1}$ induces a uniform reference $\overline{P}$, i.e., for each $X \in \mathcal{B}$

$$\overline{P}\left(X\right) = j^1_{0,X}\left(\overline{\kappa}^{-1} \circ F\tau_{\kappa(X)}\right),$$

where $\tau_{\kappa(X)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\kappa\left(X\right)$ and κ is the induced map of $\overline{\kappa}$ over $\mathcal{B}$. $\mathcal{B}$ is said to be *locally homogeneous* if every $X \in \mathcal{B}$ has a neighbourhood which is homogeneous.

Using Eq. (A.10) of the appendix A it is easy to prove the following result:

Proposition 2.1.24. *If $\mathcal{B}$ is homogeneous with respect to $\overline{Z}_0^2$ then $\overline{\omega}_{\overline{G}_0}\left(\mathcal{B}\right)$ is a non-holonomic integrable prolongation of second order. Conversely, $\overline{\omega}_{\overline{G}_0}\left(\mathcal{B}\right)$ is a second-order non-holonomic integrable prolongation implies that $\mathcal{B}$ is locally homogeneous with respect to $\overline{Z}_0^2$.*

Notice that, this result shows us that the homogeneity does not depend on the point and reference configuration but depends on the reference crystal. In section 3.2 we present a result of [52] in which we generalize the notion of homogeneity to get a definition which does not depend on the reference crystal by using an special example of groupoid: *non-holonomic groupoid of second order*. In Section 3.2 we will also introduce a treatment of homogeneity by using connections.

2.2 Groupoids

In this section we want to study the notion of *(Lie) groupoid*. Groupoids are a natural generalization of groups and may be defined as particular kind of *categories*. While groupoids were presented in 1926 by Brandt [7], categories were introduced later in 1945 by Eilenberg and McLane in [30]. In this sense, groupoids are defined as a "*small*" category such that every morphism is an isomorphism.

Adding differential structures we obtain the notion of *Lie groupoid* which was firtly introduced by Ehresmann in a collection of articles [22, 27–29] and redefined in [79] by Pradines. Roughly speaking, a Lie groupoid is defined as a groupoid satisfying that the set of morphisms and the set of objects are differentiable manifolds and the *structure maps* are differentiable.

It is remarkable that, our is not the unique application of groupoids. In fact, (Lie) groupoids are useful tools in several mathematical areas, such as *Algebraic Topology, Differential Geometry, Galois Theory, Group Theory* or *Homotopy Theory* (see [8, 83]). There are also several other research areas where groupoids are used such as *Geometric Mechanics* [16, 18, 19, 46, 66, 94] and *Quantum Mechanics* [13]. The research in which the thesis consists has resulted to be (mainly) another contribution which extends the program proposed by A. Weinstein [94] for *Continuum Mechanics*.

A good reference on groupoids is the famous book [64]. In [33] and [95] we can find a more intuitive view of this topic. The book [89] (in Spanish) is also recommendable as a rigurous introduction to groupoids.

Let us start with some examples for a smooth presentation to the concept of groupoid.

15-puzzle groupoid: Sam Lloyd claimed in 1891 that he invented the *15-puzzle*. However, due to some researches, this seems to be false. In any case, the popularity of this puzzle grew fastly (specially in Europe). The 15−puzzle consists of 15 little square blocks numbered from 1 to 15 next to a empty square enclosed in a 4×4 square box as it is shown in figure 2.1. The position of the squares exhibited in figure 2.1 is called *identity position*. Notice that the number of possible positions is exactly 16!.

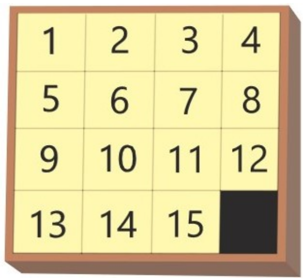


Figure 2.1: 15-puzzle

The permitted permutations of the puzzle are the sliding of the hole (one place at a time) in horizontal or vertical direction. Given any initial position, generally the goal of the game is to slide the squares around until you obtain a specified arrangement of the blocks (usually the identity position). In 1879 two American mathematicians W. W. Johnson and W. E. Story [55] achieved to prove that from any fixed initial position one cannot obtain any other random position. In fact, only half of all the possible positions can actually be obtained.
Observe that, mathematically speaking, the 15−puzzle is similar to *Rubik's Cube* because the goal is just to obtain a certain position by using only some kind of permutations. Nevertheless, as a difference with Rubik's cube, the permitted permutations depend on the initial position. For instance, when the blank square is in a corner, it can only be moved towards two positions.
A *transformation* of the 15−puzzle is given by a sequence of permitted

movements from a position to another. For example, in figure 2.2, we have a transformation resulting of moving the hole along the path $14-9-5-2-8-1$ from the iniatial position shown in this picture.

It is remarkable that any two transformations of the $15-$puzzle cannot

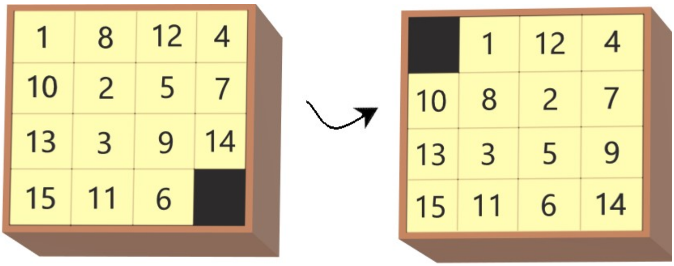


Figure 2.2: Transformation

always be composed. In particular, the composition of two transformations can only be defined when the ending position of the first transformation is equal to the starting position of the second one. For this reason, unlike the set of transformations of the Rubik's cube, the set of transformations of the $15-$puzzle does not have the structure of group. The structure of this set is the so-called *groupoid*.

Roughly speaking, the structure of groupoid is given by two sets,

- Γ : *Set of transformations of the* $15-$*puzzle.*
- M : *Set of positions of the* $15-$*puzzle.*

and a family of *structure maps* given by

- **Source and target maps**
  Source and target maps are given by two maps $\alpha, \beta : \Gamma \to M$ such that for any $g \in \Gamma$, $\alpha(g)$ (resp. $\beta(g)$) is the starting position (resp. the ending position) of the transformation g.

- **section of identities**
  The *identity map* consists of a map $\epsilon : M \to \Gamma$ satisfying that for any position $x \in M$, $\epsilon(x)$ is the identity permutation of x, i.e., the blank square is not moved.

- **Inversion map**
  The *inversion map* is a map $i : \Gamma \to \Gamma$ where for each transformation $g \in \Gamma$, $i(g)$ is the opposite transformation. For example, for the transformation given in 2.2, the opposite transformation is simply $1-8-2-5-9-14$ (exchanging the ending positions by the starting positions).

- **Composition law**
  By composing transformations we obtain a map $\cdot : \Gamma_{(2)} \to \Gamma$ where $\Gamma_{(2)}$ is just a subset of $\Gamma \times \Gamma$ given by the composable transformation.

These maps satisfy some properties such as the associativity of the composition which turn this structure into a groupoid called the 15−*puzzle groupoid*. The result of that for any two arbitrary positions generally there is not a transformation joining these two position is translated in the language of groupoid as the 15−puzzle groupoid is not *transitive*.

1-jets groupoid: Fixing a manifold M we consider the set, denoted by $\Pi^1(M,M)$, of all linear isomorphisms $L_{x,y} : T_xM \to T_yM$ for any $x,y \in M$. Any linear isomorphism $L_{x,y}$ has associated with it the points x and y of M. Denoting x by $\alpha(L_{x,y})$ and y by $\beta(L_{x,y})$ we can construct two maps $\alpha, \beta : \Pi^1(M,M) \to M$ which are the source and targent maps respectively. Notice that the isomorphism $L_{x,y}$ can be composed with another element $G_{z,t}$ of $\Pi^1(M,M)$ if, and only if

$$\alpha(G_{z,t}) = z = y = \beta(L_{x,y}).$$

So, as a difference with groups, the composition defines a partial multiplication on $\Pi^1(M,M)$. In fact, the domain of the multiplication is given by the set $\Pi^1(M,M)_{(2)}$ consisting of the elements $(G_{z,t}, L_{x,y}) \in \Pi^1(M,M) \times \Pi^1(M,M)$ such that $\alpha(G_{z,t}) = \beta(L_{x,y})$.

It is important to remark that the multiplication has similar properties as the multiplication of a group. Indeed, for each point $x \in M$ there exists the identity isomorphism $Id_{x,x} : T_x M \to T_x M$ which satisfy that

$$L_{x,y} Id_{x,x} = L_{x,y}, \quad Id_{x,x} G_{z,x} = G_{z,x},$$

for any two elements $L_{x,y}$ and $G_{z,x}$ of $\Pi^1(M, M)$ such that the above compositions are defined. Hence, the identities $Id_{x,x}$ generate the section of identities and act as unities for the partial multiplication in $\Pi^1(M, M)$. Finally, any isomorphism $L_{x,y}$ has an inverse $L_{x,y}^{-1}$ satisfying that

$$L_{x,y} L_{x,y}^{-1} = Id_{y,y}, \; L_{x,y}^{-1} L_{x,y} = Id_{x,x}.$$

This groupoid is called *1-jets groupoid on M* and it will be properly studied in examples 2.2.9 and 2.2.22. The $1-$jets groupoids $\Pi^1(M, M)$ will have a great importance in the development of the memory.

The properties above presented can be written in a more abstract and rigurous way as follows:

Definition 2.2.1. Let M be a set. A *groupoid* over M is given by a set Γ provided with the maps $\alpha, \beta : \Gamma \to M$ (*source map* and *target map* respectively), $\epsilon : M \to \Gamma$ (*section of identities*), $i : \Gamma \to \Gamma$ (*inversion map*) and $\cdot : \Gamma_{(2)} \to \Gamma$ (*composition law*) where for each $k \in \mathbb{N}$, $\Gamma_{(k)}$ is given by k points $(g_1, \ldots, g_k) \in \Gamma \times \overset{k)}{\ldots} \times \Gamma$ such that $\alpha(g_i) = \beta(g_{i+1})$ for $i = 1, \ldots, k-1$. It satisfy the following properties:

(1) α and β are surjective and for each $(g, h) \in \Gamma_{(2)}$,

$$\alpha(g \cdot h) = \alpha(h), \quad \beta(g \cdot h) = \beta(g).$$

(2) Associative law with the composition law, i.e.,

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k, \; \forall (g, h, k) \in \Gamma_{(3)}.$$

(3) For all $g \in \Gamma$,

$$g \cdot \epsilon(\alpha(g)) = g = \epsilon(\beta(g)) \cdot g.$$

In particular,

$$\alpha \circ \epsilon \circ \alpha = \alpha, \quad \beta \circ \epsilon \circ \beta = \beta.$$

Since α and β are surjetive we get

$$\alpha \circ \epsilon = Id_M, \quad \beta \circ \epsilon = Id_M,$$

where Id_M is the identity at M.

(4) For each $g \in \Gamma$,

$$i(g) \cdot g = \epsilon(\alpha(g)), \quad g \cdot i(g) = \epsilon(\beta(g)).$$

Then,

$$\alpha \circ i = \beta, \quad \beta \circ i = \alpha.$$

These maps will be called *structure maps*. Furthermore, we will denote this groupoid by $\Gamma \rightrightarrows M$.

If Γ is a groupoid over M, then M is also denoted by $\Gamma_{(0)}$ and it is often identified with the set $\epsilon(M)$ of identity elements of Γ. Γ is also denoted by $\Gamma_{(1)}$. Following the notation of the theory of categories (see Remark 2.2.3), the elements of M are called *objects* and the elements of Γ are called *morpishms*. The map $(\alpha, \beta) : \Gamma \to M \times M$ is called the *anchor map* and the space of sections of the anchor map is denoted by $\Gamma_{(\alpha,\beta)}(\Gamma)$. For any $g \in \Gamma$ the image $i(g)$ by the inversion map is denoted by g^{-1}.

Now, we define the morphisms of the category of groupoids.

Definition 2.2.2. If $\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ are two groupoids then a *morphism of groupoids* from $\Gamma_1 \rightrightarrows M_1$ to $\Gamma_2 \rightrightarrows M_2$ consists of two maps $\Phi : \Gamma_1 \to \Gamma_2$ and $\phi : M_1 \to M_2$ such that for any $g_1 \in \Gamma_1$

$$\alpha_2(\Phi(g_1)) = \phi(\alpha_1(g_1)), \qquad \beta_2(\Phi(g_1)) = \phi(\beta_1(g_1)), \tag{2.23}$$

where α_i and β_i are the source and the target map of $\Gamma_i \rightrightarrows M_i$ respectively, for $i = 1, 2$, and preserves the composition, i.e.,

$$\Phi(g_1 \cdot h_1) = \Phi(g_1) \cdot \Phi(h_1), \ \forall (g_1, h_1) \in \Gamma_{(2)}.$$

We will denote this morphism as Φ.

Observe that, as a consequence, Φ preserves the identities, i.e., denoting by ϵ_i the section of identities of $\Gamma_i \rightrightarrows M_i$ for $i = 1, 2$,

$$\Phi \circ \epsilon_1 = \epsilon_2 \circ \phi.$$

Then, using Eq. (2.23), ϕ is completely determined by Φ. The category of groupoids will be denoted by $\mathcal{G}$.

Using this definition we define a *subgroupoid* of a groupoid $\Gamma \rightrightarrows M$ as a groupoid $\Gamma' \rightrightarrows M'$ such that $M' \subseteq M$, $\Gamma' \subseteq \Gamma$ and the inclusion map is a morphism of groupoids.

Remark 2.2.3. There is a more abstract way of defining a groupoid. We can say that a groupoid is a "small" category (the class of objects and the class of morphisms are sets) in which each morphism is invertible.
If $\Gamma \rightrightarrows M$ is the groupoid, then M is the set of objects and Γ is the set of morphisms.
A groupoid morphism is a functor between these categories which is a more natural definition.
We could even find another definition of groupoid given by S. Zakrzewski in [96, 97]. Let X, Y be two sets. A *relation* r *from* X *to* Y is a triple $(Gr(r), X, Y)$ with $Gr(r) \subseteq X \times Y$. A relation from X to Y will be denoted by $r : X \to Y$.
For a relation $r : X \to Y$ we will define its *transportation* $r^T : Y \to X$ by

$$(y, x) \in Gr\left(r^T\right) \leftrightarrow (x, y) \in Gr(r).$$

The *domain* of r is the following set

$$D(r) := \{x \in X \ : \ \exists y \in Y \ \ (x, y) \in Gr(r)\},$$

and the *image*,

$$Im(r) := \{y \in Y \ : \ \exists x \in X \ \ (x, y) \in Gr(r)\}.$$

A *composition* of relations $r : X \to Y$ and $s : Z \to X$ is the relation $rs : Z \to Y$ such that

$$Gr(rs) := \{(z, y) \ : \ \exists x \in X \, (z, x) \in Gr(s), \ (x, y) \in Gr(r)\}.$$

Thus, given a family of sets C we can form a category with these relations as morphisms and C as the set of objects.
Cartesian product is defined in a natural way: Let $r_1 : X_1 \to Y_1$ and $r_2 : X_2 \to Y_2$ then $r_1 \times r_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is given by the set $Gr\left(r_1 \times r_2\right)$ of elements $\left(x_1, x_2, y_1, y_2\right)$ such that $\left(x_1, y_1\right) \in Gr\left(r_1\right)$ and $\left(x_2, y_2\right) \in Gr\left(r_2\right)$.
Finally, a groupoid $\Gamma \rightrightarrows M$ is a quadrupole (Γ, m, i, ϵ), where Γ is a set, $m : \Gamma \times \Gamma \to \Gamma$, $\epsilon : \{1\} \to \Gamma$ (the symbol $\{1\}$ denotes a one point set) and $i : \Gamma \to \Gamma$ are relations such that

(i) $m\left(m \times Id\right) = m\left(Id \times m\right)$.

(ii) $m\left(\epsilon \times Id\right) = m\left(Id \times \epsilon\right) = Id$.

(iii) $i^2 = Id$.

(iv) Considering $\sigma : \Gamma \times \Gamma \to \Gamma \times \Gamma$ with $Gr\left(\sigma\right) := \{\left(\left(x, y\right), \left(y, x\right)\right) : \left(x, y\right) \in \Gamma \times \Gamma\}$, it satisfies that

$$im = m\sigma\left(i \times i\right).$$

(v) For all $\gamma \in \Gamma$,

$$\emptyset \neq Gr\left(m\left(i\left(\gamma\right), \gamma\right)\right) \subseteq Im\left(\epsilon\right).$$

Notice that Id on a set Z denotes the relation given by the diagonal $\triangle_Z \subset Z \times Z$. ◇

Let us now present some examples of groupoids.

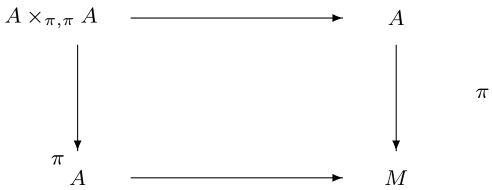
Example 2.2.4. A group is a groupoid over a point. In fact, let G be a group and e the identity element of G. Then, $G \rightrightarrows \{e\}$ is a groupoid, where the operation of the groupoid, $\cdot$, is the operation in G.

Example 2.2.5. Any set X may be regarded as a groupoid on itself with $\alpha = \beta = \epsilon = i = Id_X$ and the operation on this groupoid is given by

$$x \cdot x = x, \ \forall x \in X.$$

Note that, in this case, $X_{(2)} = \triangle_X$. We call this kind of groupoids as *base groupoids* and we will denote them as $\epsilon\left(X\right)$.

Example 2.2.6. For any set A and any map $\pi : A \to M$, we can consider the pullback space $A \times_{\pi,\pi} A$ according to the diagram



i.e.,

$$A \times_{\pi,\pi} A := \{(a_x, b_x) \in A \times A \ / \ \pi(a_x) = x = \pi(b_x)\}.$$

Then, the maps,

$$\alpha(a_x, b_x) = a_x, \quad \beta(a_x, b_x) = b_x, \ \forall (a_x, b_x) \in A \times_{\pi,\pi} A$$

$$(c_x, b_x) \cdot (a_x, c_x) = (a_x, b_x), \ \forall (c_x, b_x), (a_x, c_x) \in A \times_{\pi,\pi} A$$

$$\epsilon(a_x) = (a_x, a_x), \ \forall a_x \in A$$

$$(a_x, b_x)^{-1} = (b_x, a_x), \ \forall (a_x, b_x) \in A \times_{\pi,\pi} A$$

endow $A \times_{\pi,\pi} A$ with a structure of groupoid over A, called the *pair groupoid along* π. If $\pi = Id_A$ then this groupoid is called the *pair groupoid*.

Note that, if $\Gamma \rightrightarrows M$ is an arbitrary groupoid over M, then the anchor map $(\alpha, \beta) : \Gamma \to M \times M$ is a morphism from $\Gamma \rightrightarrows M$ to the pair groupoid of M.

The following example arises as a natural generalization of the previous one.

Example 2.2.7. Let A, M be two sets, A be a map $\pi : A \to M$ and G be a group. Then we can construct a Lie groupoid $A \times_{\pi,\pi} A \times G \rightrightarrows A$

where the source map is the second projection, the target map is the third projection and the composition law is given by the composition in G, i.e.,

$$(c_x, b_x, g) \cdot (a_x, c_x, h) = (a_x, b_x, g \cdot h),$$

for all $(c_x, b_x, g), (a_x, c_x, h) \in A \times_{\pi,\pi} A \times G$. This Lie groupoid is called *Trivial Lie groupoid along* π *with group* G. When the map π is the identity this groupoid is called *Trivial Lie groupoid on* A *with group* G.

Example 2.2.8. Let $\pi : A \to M$ be a map and $\phi : G \times A \to A$ be a left action of a group G on A which preserves the fibres, i.e.,

$$\pi \circ \phi = \pi \circ pr_2,$$

where $pr_2 : G \times A \to A$ is the projection on the second coordinate. We can construct the *transformation groupoid associated to* ϕ *along* π as follows:

The set of morphisms is $G \times A$ and the set of objects is A.

The source map and target map are given by

$$\alpha(g, a_x) = a_x, \quad \beta(g, a_x) = \phi(g, a_x),$$

for all $(g, a_x) \in G \times A$.

The operation is

$$(g, \phi(h, a_x)) \cdot (h, a_x) = (gh, a_x),$$

for all $(h, a_x), (g, \phi(h, a_x)) \in G \times A$.

The section of identities and inverse map are given by

$$\epsilon(a_x) = (e, a_x), \quad (g, a_x)^{-1} = \left(g^{-1}, \phi(g, a_x)\right),$$

for all $(g, a_x) \in G \times A$, where e is the identity element in G.

It is easy to prove that $G \times A \rightrightarrows A$ is a groupoid which will be denoted by $G \ltimes_\pi A$. For a right action, we can define the transformation groupoid analogously and we will denote this groupoid by $G \rtimes_\pi A$. In the case in which π is the identity map the groupoid is called the *transformation groupoid associated to* ϕ or simply the *transformation groupoid*

Let us take a map $\pi : A \to M$ and a left action $\phi : G \times A \to A$ of a group G on A which preserves the fibres. Then, the map

$$\begin{array}{cccc} \Phi : & G \ltimes_\pi A & \to & G \times A \times_{\pi,\pi} A \\ & (g, a_x) & \mapsto & (g, a_x, \phi(g, a_x)) \end{array}$$

is a morphism of groupoids. In fact, Φ is an isomorphism of Lie groupoids onto its image.

Example 2.2.9. Let A be a vector bundle over a manifold M. Denote the set of all vector space isomorphisms $L_{x,y} : A_x \to A_y$ for $x, y \in M$, where for each $z \in M$ A_z is the fibre of A over z, by $\Phi(A)$. We can consider $\Phi(A)$ as a groupoid $\Phi(A) \rightrightarrows M$ such that, for all $x, y \in M$ and $L_{x,y} \in \Phi(A)$,

(i) $\alpha(L_{x,y}) = x$

(ii) $\beta(L_{x,y}) = y$

(iii) $L_{y,z} \cdot G_{x,y} = L_{y,z} \circ G_{x,y},\ L_{y,z} : A_y \to A_z,\ G_{x,y} : A_x \to A_y$

This groupoid is called the *frame groupoid on* A. As a particular case, when A is the tangent bundle over M we have the example $\Pi^1(M, M)$ introduced at the beginning of the section which is called *1-jets groupoid on* M. Notice that any isomorphism $L_{x,y} : T_xM \to T_yM$ can be written as a $1-$jet $j^1_{x,y}\psi$ of a local diffeomorphism ψ from M to M (to study the formalism of $1-$jets see Appendix A).

Taking into account that any action can be seen as a particular groupoid (see example 2.2.8), it makes sense to generalize the notions of orbit and isotropy group.

Definition 2.2.10. Let $\Gamma \rightrightarrows M$ be a groupoid with α and β the source map and target map, respectively. For each $x \in M$, then

$$\Gamma^x_x = \beta^{-1}(x) \cap \alpha^{-1}(x),$$

is called the *isotropy group of* Γ at x. The set

$$\mathcal{O}(x) = \beta\left(\alpha^{-1}(x)\right) = \alpha\left(\beta^{-1}(x)\right),$$

is called the *orbit* of x, or *the orbit* of Γ through x.

Notice that the orbit of a point x consists of the points which are "*connected*" with x by a morphism in the groupoid. It is also remarkable that inside the isotropy group the composition law is globally defined and, hence, it endows the isotropy groups with a group structure.

Definition 2.2.11. If $\mathcal{O}(x) = \{x\}$, or equivalently, $\beta^{-1}(x) = \alpha^{-1}(x) = \Gamma_x^x$ then x is called a *fixed point. The orbit space of* Γ is the space of orbits of Γ on M, i.e., the quotient space of M by the equivalence relation induced by Γ: two points of M are equivalent if, and only if, they lie on the same orbit.

If $\mathcal{O}(x) = M$ for all $x \in M$, or equivalently $(\alpha, \beta) : \Gamma \to M \times M$ is a surjective map, the groupoid $\Gamma \rightrightarrows M$ is called *transitive*. If every $x \in M$ is fixed point, then the groupoid $\Gamma \rightrightarrows M$ is called *totally intransitive*. Furthermore, a subset N of M is called *invariant* if it is a union of some orbits.

Finally, the preimage of the source map α of a Lie groupoid at a point x is called $\alpha-$*fibre at* x and it is denoted by Γ_x. That of the target map β is called $\beta-$*fibre at* x and it is denoted by Γ^x.

Definition 2.2.12. Let $\Gamma \rightrightarrows M$ be a groupoid with α and β the source and target map, respectively. We may define the left translation on $g \in \Gamma$ as the map $L_g : \Gamma^{\alpha(g)} \to \Gamma^{\beta(g)}$, given by

$$h \mapsto g \cdot h.$$

We may define the right translation on g, $R_g : \Gamma_{\beta(g)} \to \Gamma_{\alpha(g)}$ similarly.

Note that, the identity map on Γ^x is given by

$$Id_{\Gamma^x} = L_{\epsilon(x)}. \tag{2.24}$$

So, for all $g \in \Gamma$, the left (resp. right) translation on g, L_g (resp. R_g), is a bijective map with inverse $L_{g^{-1}}$ (resp. $R_{g^{-1}}$).

Topological and differentiable structures could be imposed on a groupoid to get different kind of groupoids such as *topological groupoids* (see [89]). However, we will be mainly interested in *Lie groupoids.*

Definition 2.2.13. A *Lie groupoid* is a groupoid $\Gamma \rightrightarrows M$ such that Γ is a smooth manifold, M is a smooth manifold and all the structure maps are smooth. Furthermore, the source and the target map are submersions.
A *Lie groupoid morphism* is a groupoid morphism which is differentiable.

This definition permits us to contruct the category of Lie groupoids, denoted by $\mathcal{LG}$, which is in fact a subcategory of the category $\mathcal{G}$ of groupoids.

Definition 2.2.14. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A *Lie subgroupoid* of $\Gamma \rightrightarrows M$ is a Lie groupoid $\Gamma' \rightrightarrows M'$ such that Γ' and M' are submanifolds of Γ and M respectively, and the inclusion maps $i_{\Gamma'} : \Gamma' \hookrightarrow \Gamma$ $i_{M'} : M' \hookrightarrow M$ become a morphism of Lie groupoids. $\Gamma' \rightrightarrows M'$ is said to be a *reduced Lie subgroupoid* if it is transitive and $M' = M$.

It easy to check that if there exists a reduced Lie subgroupoid of a groupoid $\Gamma \rightrightarrows M$ then, $\Gamma \rightrightarrows M$ is transitive.

Observe that, taking into account that $\alpha \circ \epsilon = Id_M = \beta \circ \epsilon$, then ϵ is an injective immersion.

On the other hand, in the case of a Lie groupoid, L_g (resp. R_g) is clearly a diffeomorphism for all $g \in \Gamma$.

Note also that, for each $k \in \mathbb{N}$, $\Gamma_{(k)}$ is a pullback space given by β and the operation map on $\Gamma_{(k-1)}$. Thus, by induction, we may prove that $\Gamma_{(k)}$ is a smooth manifold for all $k \in \mathbb{N}$.

Example 2.2.15. A Lie group is a Lie groupoid over a point.

Example 2.2.16. Let M be a smooth manifold, then the base groupoid $\epsilon(M)$ (see example 2.2.5) is a Lie groupoid.

Example 2.2.17. Let $\pi : A \to M$ be a submersion. It is trivial to prove that the pair groupoid along π is a Lie groupoid.

Example 2.2.18. Let $\pi : A \to M$ be a submersion and G be a Lie group. Then, the trivial Lie groupoid along π with group G, say $A \times_{\pi,\pi} A \times G \rightrightarrows A$, is obviously a Lie groupoid.

Example 2.2.19. Let $\pi : A \to M$ be a submersion and $\phi : G \times A \to A$ be a (left) action of a Lie group G on A which preserves the fibres. Then, transformation groupoid $G \ltimes_{\pi} A$ associated to ϕ is a Lie groupoid.

Example 2.2.20. Let $\pi : P \to M$ be a principal bundle with structure group G. Denote by $\phi : G \times P \to P$ the action of G on P.

Now, suppose that $\Gamma \rightrightarrows P$ is a Lie groupoid, with $\overline{\phi} : G \times \Gamma \to \Gamma$ a free and proper action of G on Γ such that, for each $g \in G$, the pair $\left(\overline{\phi}_g, \phi_g\right)$ is an isomorphism of Lie groupoids.

We can construct a Lie groupoid $\Gamma/G \rightrightarrows M$ such that the source map, $\overline{\alpha}$, and the target map, $\overline{\beta}$, are given by

$$\overline{\beta}\left([g]\right) = \pi\left(\beta\left(g\right)\right), \quad \overline{\alpha}\left([g]\right) = \pi\left(\alpha\left(g\right)\right),$$

for all $g \in \Gamma$, α and β being the source and the target map on $\Gamma \rightrightarrows P$, respectively, and $[\cdot]$ denotes the equivalence class in the quotient space Γ/G. These kind of Lie groupoids are called *quotient Lie groupoids by the action of a Lie group.*

There is an interesting particular case of the above example.

Example 2.2.21. Let $\pi : P \to M$ be a principal bundle with structure group G and $P \times P \rightrightarrows P$ the pair groupoid. Take $\overline{\phi} : G \times (P \times P) \to P \times P$ the diagonal action of ϕ, where $\phi : G \times P \to P$ is the action of G on P.

Then it is easy to prove that $\left(\overline{\phi}_g, \phi_g\right)$ is an isomorphism of Lie groupoids and thus, we may construct the groupoid $\left(P \times P\right)/G \rightrightarrows M$. This groupoid is called *gauge groupoid* and is denoted by $Gauge\left(P\right)$.

Example 2.2.22. Let A be a vector bundle over M then the frame groupoid is a Lie groupoid (see example 2.2.9). In fact, let $\left(x^i\right)$ and $\left(y^j\right)$ be local coordinate systems on open sets $U, V \subseteq M$ and $\{\alpha_p\}$ and $\{\beta_q\}$ be local basis of sections of A_U and A_V respectively. The corresponding local coordinates $\left(x^i \circ \pi, \alpha^p\right)$ and $\left(y^j \circ \pi, \beta^q\right)$ on A_U and A_V are given by

- For all $a \in A_U$,
$$a = \alpha^p\left(a\right)\alpha_p\left(x^i\left(\pi\left(a\right)\right)\right).$$

- For all $a \in A_V$,
$$a = \beta^q (a) \beta_q \left(y^j (\pi (a)) \right).$$

Then, we can consider a local coordinate system $\Phi (A)$

$$\Phi \left(A_{U,V} \right) : \left(x^i, y^j_i, y^j_i \right),$$

where, $A_{U,V} = \alpha^{-1} (U) \cap \beta^{-1} (V)$ and for each $L_{x,y} \in \alpha^{-1} (x) \cap \beta^{-1} (y) \subseteq \alpha^{-1} (U) \cap \beta^{-1} (V)$,

- $x^i (L_{x,y}) = x^i (x)$.
- $y^j (L_{x,y}) = y^j (y)$.
- $y^j_i (L_{x,y}) = A_{L_{x,y}}$, where $A_{L_{x,y}}$ is the induced matrix of the induced map of $L_{x,y}$ by the local coordinates $\left(x^i \circ \pi, \alpha^p \right)$ and $\left(y^j \circ \pi, \beta^q \right)$.

In particular, if $A = TM$, then the $1-$jets groupoid on M, $\Pi^1 (M, M)$, is a Lie groupoid and its local coordinates will be denoted as follows

$$\Pi^1 (U, V) : \left(x^i, y^j, y^j_i \right), \tag{2.25}$$

where, for each $j^1_{x,y} \psi \in \Pi^1 (U, V)$

- $x^i \left(j^1_{x,y} \psi \right) = x^i (x)$.
- $y^j \left(j^1_{x,y} \psi \right) = y^j (y)$.
- $y^j_i \left(j^1_{x,y} \psi \right) = \dfrac{\partial \left(y^j \circ \psi \right)}{\partial x^i_{|x}}$.

Next, as an important example, we will introduce the *second-order non-holonomic groupoid.*

Example 2.2.23. Let M be a manifold and FM the frame bundle over M. So, we can consider the $1-$jets groupoid on FM, $\Pi^1(FM, FM) \rightrightarrows FM$.
Thus, we denote by $J^1(FM)$ the subset of $\Pi^1(FM, FM)$ given by the $1-$jets $j^1_{\overline{x},\overline{y}}\Psi$ of local automorphism Ψ of FM such that

$$\Psi(g \cdot v) = g \cdot \Psi(v), \ \forall v \in Dom(\Psi), \ \forall g \in Gl(n, \mathbb{R}).$$

Let (x^i) and (y^j) be local coordinate systems over two open sets $U, V \subseteq M$, the induced coordinate systems over FM (see Appendiix A) are denoted by

$$FU : (x^i, x^i_j)$$

$$FV : \left(y^j, y^j_i\right).$$

Hence, we can construct induced coordinates over $\Pi^1(FM, FM)$

$$(\alpha, \beta)^{-1}(U, V) : \left((x^i, x^i_j), \left(y^j, y^j_i\right), y^j_{,i}, y^j_{,ik}, y^j_{i,k}, y^j_{i,kl} \right),$$

where for each $j^1_{\overline{x},\overline{y}}\Psi \in \Pi^1(FU, FV) = (\alpha, \beta)^{-1}(U, V)$, we have

- $x^i\left(j^1_{\overline{x},\overline{y}}\Psi\right) = x^i(\overline{x})$
- $x^i_j\left(j^1_{\overline{x},\overline{y}}\Psi\right) = x^i_j(\overline{x})$
- $y^j\left(j^1_{\overline{x},\overline{y}}\Psi\right) = y^j(\Psi(\overline{x}))$
- $y^j_i\left(j^1_{\overline{x},\overline{y}}\Psi\right) = y^j_i(\Psi(\overline{x}))$
- $y^j_{,i}\left(j^1_{\overline{x},\overline{y}}\Psi\right) = \dfrac{\partial\left(y^j \circ \Psi\right)}{\partial x^i_{|\overline{x}}}$
- $y^j_{,ik}\left(j^1_{\overline{x},\overline{y}}\Psi\right) = \dfrac{\partial\left(y^j \circ \Psi\right)}{\partial x^i_{k\,|\overline{x}}}$

- $y^j_{i,k}\left(j^1_{\overline{x},\overline{y}}\Psi\right) = \dfrac{\partial\left(y^j_i \circ \Psi\right)}{\partial x^k}_{|\overline{x}}$

- $y^j_{i,kl}\left(j^1_{\overline{x},\overline{y}}\Psi\right) = \dfrac{\partial\left(y^j_i \circ \Psi\right)}{\partial x^k_l}_{|\overline{x}}$

Then, using these coordinates, $J^1(FM)$ can be (locally) described as follows:

$$\left(\left(x^i, x^i_j\right), \left(y^j, y^j_i\right), y^j_{,i}, 0, y^j_{i,k}, y^j_{i,kl}\right),$$

over the set $J^1(FU, FV) = J^1(FM) \cap (\alpha, \beta)^{-1}(U, V)$ where

$$y^j_{i,kl} = \left(y^j_m \left(x^{-1}\right)^m_k\right)\delta^i_l.$$

Thus, $J^1(FM)$ is a submanifold of $\Pi^1(FM, FM)$ and its induced local coordinates will be denoted by

$$J^1(FU, FV) : \left(\left(x^i, x^i_j\right), \left(y^j, y^j_i\right), y^j_{,i}, y^j_{i,k}\right). \tag{2.26}$$

Finally, restricting the structure maps we can ensure that $J^1(FM) \rightrightarrows FM$ is a reduced Lie subgroupoid of the $1-$jets groupoid over FM.

Analogously to F^2M, we may construct $j^1(FM)$ as the set of the $1-$jets of the form $j^1_{\overline{x},\overline{y}}F\psi$, where $\psi : M \to M$ is a local diffeomorphism. Let $\left(x^i\right)$ be a local coordinate system on M; then, restricting the induced local coordinates given in Eq. (2.26) to $j^1(FM)$ we have that

$$y^j_i = y^j_{,l}x^l_i \quad ; \quad y^j_{i,k} = y^j_{k,i}.$$

We deduce that $j^1(FM) \rightrightarrows FM$ is a reduced Lie subgroupoid of the $1-$jets groupoid over FM and we denoted the coordinates on $j^1(FM)$ by

$$j^1(FU, FV) : \left(\left(x^i, x^i_j\right), \left(y^j, y^j_i\right), y^j_{i,k}\right), \quad y^j_{i,k} = y^j_{k,i}. \tag{2.27}$$

Now, we will work with a quotient space of $J^1(FM)$ (resp. $j^1(FM)$) which will be our *non-holonomic groupoid of second order* (resp. *holonomic groupoid of second order*).

We consider the following left action of $Gl(n,\mathbb{R})$ over $J^1(FM)$,

$$\begin{array}{rccl} \Phi: & Gl(n,\mathbb{R}) \times J^1(FM) & \to & J^1(FM) \\ & \left(g, j^1_{\overline{x},\overline{y}}\Psi\right) & \mapsto & j^1_{\overline{x}\cdot g,\overline{y}\cdot g}\Psi. \end{array} \tag{2.28}$$

Thus, for each $g \in Gl(n,\mathbb{R})$ the pair (R_g, Φ_g) (where L is the natural right action of $Gl(n,\mathbb{R})$ over FM) is a Lie groupoid automorphism. Therefore, we can consider the quotient Lie groupoid by this action $\tilde{J}^1(FM) \rightrightarrows M$ which is called *second-order non-holonomic groupoid over* M.
We will denote the structure maps of $\tilde{J}^1(FM)$ by $\overline{\alpha}$ and $\overline{\beta}$ (source and target maps respectively), $\overline{\epsilon}$ (identities map) and $\overline{i}$ (inversion map). The elements of $\tilde{J}^1(FM)$ are denoted by $j^1_{x,y}\Psi$ with $x, y \in M$ and $\overline{\alpha}\left(j^1_{x,y}\Psi\right) = x$ and $\overline{\beta}\left(j^1_{x,y}\Psi\right) = y$.
Then, the induced local coordinates are given by

$$\tilde{J}^1(FU,FV) : \left(\left(x^i\right), \left(y^j, y^j_i\right), y^j_{,i}, y^j_{i,k}\right), \tag{2.29}$$

where $\tilde{J}^1(FU,FV) = \left(\overline{\alpha},\overline{\beta}\right)^{-1}(U,V)$. Considering e_{1x} as the 1−jet through $x \in M$ which satisfies that $x^i_j(e_{1x}) = \delta^i_j$ for all i,j, for each $j^1_{x,y}\Psi \in \tilde{J}^1(FM)$ we have

- $x^i\left(j^1_{x,y}\Psi\right) = x^i(x)$
- $y^j\left(j^1_{x,y}\Psi\right) = y^j(y)$
- $y^j_i\left(j^1_{x,y}\Psi\right) = y^j_i(\Psi(e_{1x}))$.
- $y^j_{,i}\left(j^1_{x,y}\Psi\right) = \dfrac{\partial\left(y^j \circ \Psi\right)}{\partial x^i_{|e_{1x}}}$

- $y^j_{i,k}\left(j^1_{x,y}\Psi\right) = \dfrac{\partial\left(y^j_i \circ \Psi\right)}{\partial x^k}_{|e_{1x}}$

Observe that we can restrict the action Φ to an action of $Gl\left(n, \mathbb{R}\right)$ over $j^1\left(FM\right)$. So, by quotienting, we can build a reduced subgroupoid of $\tilde{J}^1\left(FM\right) \rightrightarrows M$ which is denoted by $\tilde{j}^1\left(FM\right) \rightrightarrows M$ and is called *second-order holonomic groupoid over M*. Finally, by restriction, the local coordinates on $j^1\left(FM\right)$ are given by

$$\tilde{j}^1\left(FU, FV\right) : \left(\left(x^i\right), \left(y^j, y^j_i\right), y^j_{i,k}\right), \quad y^j_{i,k} = y^j_{k,i}. \tag{2.30}$$

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Using that β, α are submersions, we have that the $\beta-$fibres and the $\alpha-$fibres are closed submanifolds of Γ. Moreover, the following lemma will be useful to prove some fundamental results over Lie groupoids.

Lemma 2.2.24. *Let $\phi : G \times M \to M$ be a free (left) action of a Lie group G on a manifold M. The following conditions are equivalent.*

(i) For any $x \in M$ there exists an embedded submanifold N_x with $x \in N_x$ such that $G \times N_x \to M$ given by the restriction of the action of G is an open embedding.

(ii) There is a smooth (perhaps non-Hausdorff) structure on M/G such that the quotient projection $M \to M/G$ is a principal $G-$bundle.

(iii) There exist a (perhaps non-Hausdorff) manifold X and a smooth map $f : M \to X$ which is constant on the $G-$orbits and satisfies

$$Ker\left(T_x f\right) = T_{(e,x)}\phi\left(\{0\} \times T_e G\right),$$

for all $x \in M$

Proof. If (i) holds, then for each $x \in M$ the restriction of the quotient projection $N_x \to M/G$ is a topological open embedding (note that this map is trivially injective), and we may define a smooth structure on M/G such that this map is a smooth open embedding. Therefore (i) implies (ii). Note that (iii) follows directly from (ii) ($X = M/G$ and f is the quotient

map). So we only need to prove that (iii) implies (i).
Take any $x \in M$, and choose a submersion $h : V \to \mathbb{R}^k$ defined on an open neighbourhood V of $f(x)$ in X such that $Ker\left(T_{f(x)}h\right)$ is complementary to $T_x f(T_x M)$. Next, choose a small transversal section N_x of the foliation of M given by the connected components of the $G-$orbits, with $x \in N_x$ and $f(N_x) \subseteq V$. Now, by construction, $T_x\left(h \circ f_{|N_x}\right)$ is an isomorphism, so we may shrink N_x if necessary so that

$$h \circ f_{|N_x},$$

is an open embedding. In particular, f is injective on N_x. Since f is also constant along the $G-$orbits, it follows that each $G-$orbit intersects N_x in at most one point. Since N_x is transversal to the $G-$orbits, this proves (i). □

Thus, we may prove the following results.

Lemma 2.2.25. *If $\Gamma \rightrightarrows M$ is a Lie groupoid, then for all $x, y \in M$ $\Gamma_x \cap \Gamma^y$ is a closed submanifold of Γ.*

Proof. First, we may construct the distribution $\mathcal{H}$ on Γ, given by

$$g \mapsto \mathcal{H}_g = Ker\left(T_g\alpha\right) \cap Ker\left(T_g\beta\right), \ \forall g \in \Gamma.$$

Now, consider the left translation

$$L_g : \Gamma^{\alpha(g)} \to \Gamma^{\beta(g)},$$

which is a diffeomorphism between $t-$fibers. Observe that, for any $h \in \Gamma^{\alpha(g)}$, $\mathcal{H}_g$ is a subspace of $T_g\Gamma^{\alpha(g)} = Ker\left(T_g\beta\right)$. Using that $\alpha \circ L_g = \alpha_{|\Gamma^{\alpha(g)}}$, it follows that

$$T_{\epsilon(\alpha(g))}L_g\left(\mathcal{H}_{\epsilon(\alpha(g))}\right) = \mathcal{H}_g.$$

In addition, any basis $v_1, \ldots, v_k$ of $\mathcal{H}_{\epsilon(\alpha(g))}$ can be extended to a global frame $X_1, \ldots, X_k$ of $\mathcal{H}_{\Gamma^{\alpha(g)}}$ by

$$X_i(g) = T_{\epsilon(\alpha(g))}L_g\left(v_i\right).$$

In this way, the restriction of $\mathcal{H}$ to the $\beta-$fibers is a locally finitely generated smooth distribution. It is involutive because it is exactly the kernel of the derivative of the map $\beta_{|\Gamma_{\alpha(g)}}$. Hence, using Hermann's theorem B.0.22, it defines a foliation $\mathcal{F}_x$ of Γ^x (which is parallelizable by the frame $X_1, \ldots, X_k$). The leaves of $\mathcal{F}_x$ are exactly the connected components of the $\alpha-$fibres of $\beta_{|\Gamma^x}$. So these fibres are closed manifolds. □

Immediately we have the following corollary

Corollary 2.2.26. *If $\Gamma \rightrightarrows M$ is a Lie groupoid, then for any $x \in M$, the isotropy group Γ_x^x is a Lie group.*

Now, we can construct a left action of Γ_x^x on $\beta^{-1}(x)$, $\phi : \Gamma_x^x \times \Gamma^x \to \Gamma^x$, given by

$$\phi(g,h) = L_g(h), \ \forall (g,h) \in \Gamma_x^x \times \Gamma^x.$$

From this action, we can give structure of smooth manifold to the orbits as follows

Lemma 2.2.27. *If $\Gamma \rightrightarrows M$ is a Lie groupoid, then for all $x \in M$ there is a natural structure of a smooth manifold on the orbit $\mathcal{O}(x)$ making $\alpha_{|\Gamma^x} : \Gamma^x \to \mathcal{O}(x)$ into a principal Γ_x^x-bundle*

Proof. As we have seen, the Lie group Γ_x^x acts smoothly and freely on Γ^x from the left, and it acts transitively along the manifolds $\Gamma_y \cap \Gamma^x$. Note that the condition (*iii*) of lemma 2.2.24 is fulfilled by the map $\alpha_{|\Gamma^x}$, so the proposition implies that there is a natural structure of a smooth manifold on the orbit $\mathcal{O}(x)$ making $\alpha_{|\Gamma^x} : \Gamma^x \to \mathcal{O}(x)$ into a principal Γ_x^x-bundle. The fact that M is Hausdorff implies that $\mathcal{O}(x)$ is also Hausdorff. □

Observe that, taking into account that $\alpha_{|\Gamma^x} : \Gamma^x \to \mathcal{O}(x)$ is a principal Γ_x^x-bundle, we may consider $Gauge(\Gamma^x)$ (see example 2.2.21). So, as a corollary, we have the following result.

Corollary 2.2.28. *If $\Gamma \rightrightarrows M$ is transitive, $\Gamma \cong Gauge(\Gamma^x)$.*

Proof. Consider the map

$$\begin{array}{rccc} \Phi : & \Gamma^x \times \Gamma^x / \Gamma_x^x & \to & \Gamma \\ & [(g,h)] & \mapsto & g^{-1}h \end{array}.$$

Suppose that $[(g,h)] = [(g',h')]$, then there exists $k \in \Gamma$ such that $g' = kg$ and $h' = kh$. Therefore,

$$\left(g'\right)^{-1} h' = (kg)^{-1}(kh) = g^{-1}h.$$

So, Φ is well defined. Furthermore, composing with the quotient projection map, we get that Φ is a smooth map.
Also, let $[(g,h)], [(g',h')] \in \Gamma^x \times \Gamma^x / \Gamma_x^x$ such that $\Phi([(g,h)]) = g^{-1}h = (g')^{-1} h' = \Phi([(g',h')])$. Then, taking $k = g'(g)^{-1}$, we have

$$kg = g', \qquad kh = h',$$

i.e.,

$$[(g,h)] = [\left(g',h'\right)].$$

On the other hand, let $k \in \Gamma$ with $\beta(k) = y$. Using that $\mathcal{O}(x) = M$, there exists $g \in \Gamma^x$ such that $\alpha(g) = y$. Hence, $(gk, g) \in \Gamma^x \times \Gamma^x$ and

$$\Phi([(g, gk)]) = k.$$

In this way, we haved proved that Φ is a bijective map. So, it is clear that Φ is a Lie groupoid isomorphism over the identity. □

It is important to remark the importance of this result. In fact, we have proved that the only transitive Lie groupoids are the Gauge groupoids presented in example 2.2.21.

2.3 Algebroids

The notion of *Lie algebroid* was introduced by J. Pradines in 1966 [79] as an infinitesimal version of Lie groupoid and for this reason the first name of this objet was *infinitesimal groupoid.* To study this notion we also refer to [64]. Let us present a basic example of Lie algebroid to introduce the reader to the notion.

Tangent bundle: Let M be a manifold. Then, the tangent bundle TM of M defines what is known as *Lie algebroid.* Consider the canonical projection $\pi_M : TM \to M$ of the tangent bundle of M. Then, the space of sections of π_M is the module of vector fields $\mathfrak{X}(M)$ on M and we have the following structure,

- **Anchor:** The identity on TM is a morphism of vector bundles from the domain of π_M, i.e. TM, to TM which, in general, will be called the *anchor map.*

- **Lie bracket:** The Lie bracket $[\cdot,\cdot]$ of vector fields is a bracket on $\mathfrak{X}(M)$ such that $(\mathfrak{X}(M),[\cdot,\cdot])$ is a Lie algebra.

- **Leibniz rule:** It satisfies the following property,

$$[\Theta_1, f\Theta_2] = f[\Theta_1,\Theta_2] + \Theta_1(f)\,\Theta_2,$$

for all $\Theta_1,\Theta_2 \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$.

These properties turns the tangent bundle into a Lie algebroids. The definition of Lie algebroid will be properly exposed in definition 2.3.1.
Consider now the pair Lie groupoid $M\times M \rightrightarrows M$ on M (see examples 2.2.6 and 2.2.17). A *left-invariant vector field* on the pair groupoid is simply a vector field Θ on $M\times M$ such that

$$\Theta(g\cdot h) = T_g L_h(\Theta(g)),$$

for all $g,h \in M\times M$ satisfying that $\alpha(g)=\beta(h)$. Notice that, by definition 2.2.12, we have that Θ should be tangent to the $\beta-$fibres. Therefore, taking into account the left invariance, we have that the space of left-invariant vector fields $M\times M$ can be identified with the space of vector fields on M. This identication is, in fact, a Lie algebra morphism (the structure of Lie bracket is preserved).
Now, let $A(M\times M)$ be the vector bundle on M such that the fibre $A(M\times M)_x$ at some $x\in M$ is given by the tangent space of the $\beta-$fibre at the identity morphism $\epsilon(x)=(x,x)$. This vector bundle will be what is called *the infinitesimal version of* $M\times M$.
Restricting the left-invariant vector fields on $M\times M$ to the identity morphism we obtain an isomorphism from the space of left-invariant vector fields on $M\times M$ to the space of sections $\Gamma(A(M\times M))$ of $A(M\times M)$. This isomorphism endows the space $\Gamma(A(M\times M))$ with a structure of Lie algebra which clearly satisfy the Leibniz rule. Hence, the infinitesimal version of the pair groupoid $M\times M\rightrightarrows M$ on M will be (isomorphic to) the Lie algebroid structure of the tangent bundle on M.
Again, this construction will be explained with more rigurosity in 2.3.

Definition 2.3.1. A *Lie algebroid over* M is a triple $(A \to M, \sharp, [\cdot,\cdot])$, where $\pi : A \to M$ is a vector bundle together with a vector bundle morphism $\sharp : A \to TM$, called the *anchor*, and a Lie bracket $[\cdot,\cdot]$ on the space of sections, such that the Leibniz rule holds

$$[\Lambda_1, f\Lambda_2] = f[\Lambda_1,\Lambda_2] + \sharp(\Lambda_1)(f)\,\Lambda_2, \tag{2.31}$$

for all $\Lambda_1, \Lambda_2 \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$.
A is *transitive* if $\sharp$ is surjective and *totally intransitive* if $\sharp \equiv 0$. Also, A is said to be *regular* if $\sharp$ has constant rank.

Looking at $\sharp$ as a $\mathcal{C}^\infty(M)$-module morphism from $\Gamma(A)$ to $\mathfrak{X}(M)$, for each section $\Lambda_1 \in \Gamma(A)$ we are going to denote $\sharp(\Lambda_1)$ by $\Lambda_1^\sharp$. Next, let us show the following fundamental property:

Lemma 2.3.2. *If* $(A \to M, \sharp, [\cdot,\cdot])$ *is a Lie algebroid, then the anchor map is a morphism of Lie algebras, i.e.*

$$[\Lambda_1,\Lambda_2]^\sharp = \left[\Lambda_1^\sharp, \Lambda_2^\sharp\right], \ \forall \Lambda_1,\Lambda_2 \in \Gamma(A). \tag{2.32}$$

Proof. Let $\Lambda_1, \Lambda_2 \in \Gamma(A)$. By the Jacobi identity, for any section $\gamma \in \Gamma(A)$ and any function $f \in \mathcal{C}^\infty(M)$, we have

$$0 = [[\Lambda_1,\Lambda_2], f\gamma] + [[f\gamma,\Lambda_1],\Lambda_2] + [[\Lambda_2, f\gamma],\Lambda_1]. \tag{2.33}$$

Now, using the Leibniz rule,

- $[[\Lambda_1,\Lambda_2], f\gamma] = f[[\Lambda_1,\Lambda_2],\gamma] + [\Lambda_1,\Lambda_2]^\sharp(f)\,\gamma.$
- $[[f\gamma,\Lambda_1],\Lambda_2] = f[\Lambda_2,[\Lambda_1,\gamma]] + \Lambda_2^\sharp(f)[\Lambda_1,\gamma] + \Lambda_1^\sharp(f)[\Lambda_2,\gamma] + \Lambda_2^\sharp\left(\Lambda_1^\sharp(f)\right)\gamma.$
- $[[\Lambda_2, f\gamma],\Lambda_1] = -f[\Lambda_1,[\Lambda_2,\gamma]] - \Lambda_1^\sharp(f)[\Lambda_2,\gamma] - \Lambda_2^\sharp(f)[\Lambda_1,\gamma] - \Lambda_1^\sharp\left(\Lambda_2^\sharp(f)\right)\gamma.$

If we replace these equalities in Eq. (2.33), we have

$$\begin{aligned} 0 &= [\Lambda_1,\Lambda_2]^\sharp(f)\,\gamma + \Lambda_2^\sharp\left(\Lambda_1^\sharp(f)\right)\gamma - \Lambda_1^\sharp\left(\Lambda_2^\sharp(f)\right)\gamma \\ &= [\Lambda_1,\Lambda_2]^\sharp(f)\,\gamma - \left[\Lambda_2^\sharp,\Lambda_1^\sharp\right](f)\,\gamma \end{aligned}$$

for any $\gamma \in \Gamma(A)$ and for any $f \in \mathcal{C}^\infty(M)$. Thus, we conclude that

$$[\Lambda_1, \Lambda_2]^\sharp = \left[\Lambda_1^\sharp, \Lambda_2^\sharp\right], \ \forall \Lambda_1, \Lambda_2 \in \Gamma(A).$$

□

Remark 2.3.3. Eq. (2.32) is often considered as a part of the definition of a Lie algebroid though, as we have seen, it is a consequence of the other conditions.
◇

Let x be a point at M and A_x be the fibre of the Lie algebroid A at x. Then, we may define a linear map $\sharp_x : A_x \to T_xM$ as the restriction of the anchor $\sharp$ to the fibres A_x and T_xM.

Definition 2.3.4. The *isotropy algebra* of the Lie algebroid A at the point $x \in M$ is the Lie algebra $\left(Ker(\sharp_x), [\cdot,\cdot]_x\right)$, the Lie bracket is given by

$$[\Lambda_{1x}, \Lambda_{2x}]_x = [\Lambda_1, \Lambda_2](x),$$

for any two sections $\Lambda_1, \Lambda_2 \in \Gamma(A)$ such that

$$\Lambda_i(x) = \Lambda_{ix}, \quad i = 1, 2.$$

It is not hard to prove (see example 2.3.6) that the bracket $[\cdot,\cdot]_x$ is well defined and, hence, defines a Lie algebra structure on the vector space $Ker(\sharp_x)$.
An important remark is that the Lie algebra structure on sections is of local type i.e. $[\Lambda_1, \Lambda_2](x)$ will depend on Λ_2 (therefore, on Λ_1 too) around x only, $\forall x \in M$. Indeed, if $\Lambda_2, \widehat{\Lambda_2} \in \Gamma(A)$ with

$$\Lambda_{2|U} = \widehat{\Lambda_2}_{|U},$$

for an open neighbourhood U of x, taking $f \in \mathcal{C}^\infty(M)$ such that $supp(f) \subseteq U$, $f \equiv 1$ on a compact neighbourhood $V_x \subset U$ of x, then $f\Lambda_2 = f\widehat{\Lambda_2}$ on M. Using the Leibniz rule

$$\begin{aligned}[\Lambda_1, f\Lambda_2](x) &= [\Lambda_1, \Lambda_2](x) + \Lambda_1^\sharp(x)(f)\Lambda_2(x) \\ &= [\Lambda_1, \Lambda_2](x),\end{aligned}$$

since $\Lambda_1^\sharp(x)(f) = 0$ (f is constant on a neighbourhood of x). Thus,

$$\begin{aligned}\left[\Lambda_1, \widehat{\Lambda_2}\right](x) &= \left[\Lambda_1, f\widehat{\Lambda_2}\right](x) \\ &= \left[\Lambda_1, f\Lambda_2\right](x) \\ &= \left[\Lambda_1, \Lambda_2\right](x).\end{aligned}$$

Finally, from skew-symmetry the result is proved.
As a consequence, the restriction of a Lie algebroid over M to a open subset of M is again a Lie algebroid. Taking local coordinates (x^i) on M and a local basis of sections of A, $\{\Lambda_p\}$, the corresponding local coordinates $(x^i \circ \pi, y^p)$ on A, satisfy

$$a = y^p(a)\,\Lambda_p\left(x^i(\pi(a))\right), \ \forall a \in \pi^{-1}(U).$$

Such coordinates determine local functions $\sharp_p^i$, C_{pq}^r on M which contain the local information of the Lie algebroid structure, and accordingly they are called *the structure functions of the Lie algebroid.* They are given by

$$\Lambda_p^\sharp = \sharp_p^i \frac{\partial}{\partial x^i},$$

$$[\Lambda_p, \Lambda_q] = C_{pq}^r \Lambda_r.$$

Imposing Eq. (2.32) and the Jacobi identity over the local basis $\{\Lambda_p\}$, we get the following equations

$$C_{pq}^r \sharp_r^i = \left(\sharp_p^r \frac{\partial \sharp_q^i}{\partial x^r} - \sharp_q^r \frac{\partial \sharp_p^i}{\partial x^r}\right),$$

$$\oint_{pqr} \sharp_p^t \frac{\partial C_{qr}^s}{\partial x^t} + C_{pt}^s C_{qr}^t = 0,$$

for all i, p, q, where $\oint_{ijk} a_{ijk}$ means the cyclic sum $a_{ijk} + a_{kij} + a_{jki}$. These equations are usually called *structure equations.*

Now, we will give some examples of Lie algebroids

Example 2.3.5. Any Lie algebra is a Lie algebroid over a single point. Indeed, identifying $\Gamma(\mathfrak{g})$ with $\mathfrak{g}$, the Lie bracket on sections is simply the Lie algebra bracket and the anchor map is the trivial one.

This kind of Lie algebroid is a particular case of the following example.

Example 2.3.6. Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid where $\sharp \equiv 0$. Then, the Lie bracket on $\Gamma(A)$ is a point-wise Lie bracket, that is, the restriction of $[\cdot,\cdot]$ to the fibres induces a Lie algebra structure on each of them. More precisely, using that $\sharp \equiv 0$, the Leibniz rule is just $\mathcal{C}^\infty(M)-$linearity, i.e.,

$$[\Lambda_1, f\Lambda_2] = f[\Lambda_1, \Lambda_2], \ \forall f \in \mathcal{C}^\infty(M), \ \forall \Lambda_1, \Lambda_2 \in \Gamma(A). \tag{2.34}$$

Consider $x \in M$ and $\Lambda_2, \widehat{\Lambda_2} \in \Gamma(A)$ such that

$$\Lambda_2(x) = \widehat{\Lambda_2}(x).$$

Let $\{\gamma_1, \ldots, \gamma_k\}$ be a basis of local sections. Then, around x, we have

$$\Lambda_2 - \widehat{\Lambda_2} = f_i \gamma_i,$$

with $f_i(x) = 0$, for all $i \in \{1, \ldots, k\}$. From Eq. (2.34) for each $\Lambda_1 \in \Gamma(A)$ we have

$$\left[\Lambda_1, \Lambda_2 - \widehat{\Lambda_2}\right](x) = f_i(x)[\Lambda_1, \gamma_i](x) = 0,$$

Therefore,

$$[\Lambda_1, \Lambda_2](x) = \left[\Lambda_1, \widehat{\Lambda_2}\right](x).$$

Finally, skew-symmetry allows us to prove that the value of $[\Lambda_1, \Lambda_2]$ in a point $x \in M$ depends only on $\Lambda_1(x)$ and $\Lambda_2(x)$. These kind of Lie algebroids (with $\sharp \equiv 0$) are called *Lie algebra bundles*. Note that the Lie algebra structures on the fibres are not necessary isomorphic to each other.

Example 2.3.7. Following the initial example, for any smooth manifold M, the tangent bundle of M, TM, is a Lie algebroid: the anchor map is the identity map and the Lie bracket is the usual Lie bracket of vector fields. This is called the *tangent algebroid* of M.

Example 2.3.8. Let M be a manifold and $\mathfrak{g}$ be a Lie algebra. We can construct a Lie algebroid structure over the vector bundle $A = TM \oplus (M \times \mathfrak{g}) \to M$ such that

(i) The anchor $\sharp : TM \oplus (M \times \mathfrak{g}) \to TM$ is the projection.

(ii) Lie algebra structure over the space of sections is given by:

$$[X \oplus f, Y \oplus g] = [X, Y] \oplus \{X(g) - Y(f) + [f, g]\},$$

for all $X \oplus f, Y \oplus g \in \Gamma(A)$.

This Lie algebroid is called the *Trivial Lie algebroid on* M *with structure algebra* $\mathfrak{g}$.

Example 2.3.9. If M is a manifold and D is an involutive subbundle of TM, then D is a Lie algebroid over M, where the anchor is the inclusion $i : D \to TM$ and the bracket is the restriction of the Lie bracket of vector fields. Thus, let $\mathcal{F}$ be a regular foliation of M. Then the *tangent algebroid* of $\mathcal{F}$ is the subbundle of TM, $T\mathcal{F}$, consisting of tangent spaces to $\mathcal{F}$ with the usual Lie bracket, and the inclusion map as the anchor.

Note that, since $\mathcal{F}$ is regular, $T\mathcal{F}$ is a subbundle of TM, and its sections are vector fields tangent to $\mathcal{F}$. Moreover, $T\mathcal{F}$ being regular and integrable, implies that it is involutive and, as a consequence, the Lie bracket of two vector fields tangent to $\mathcal{F}$ is again a vector field tangent to $\mathcal{F}$.

Example 2.3.10. Let M be a smooth manifold, $\mathfrak{g}$ be a Lie algebra and $\xi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra morphism (i.e. $\xi : \mathfrak{g} \to \mathfrak{X}(M)$ is an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M). It is possible to associate to it the following *transformation algebroid*:

- **Vector bundle:** The vector bundle is the trivial bundle $\mathfrak{g} \times M \to M$
- **Anchor:** The anchor map is $\sharp : \mathfrak{g} \times M \to TM$ such that

$$\sharp(u, x) = \xi(u)(x).$$

So, the anchor map is the fixed Lie algebra morphism ξ.

- **Lie bracket:** The Lie bracket is given by

$$[\Lambda_1, \Lambda_2](z) = [\Lambda_1(z), \Lambda_2(z)]_{\mathfrak{g}} + + (\xi(\Lambda_1(z)))_z(\Lambda_2) - (\xi(\Lambda_2(z)))_z(\Lambda_1), \tag{2.35}$$

  where we are identifying $\Lambda_1 \in \Gamma(\mathfrak{g} \times M)$ with a smooth map $\Lambda_1 : M \to \mathfrak{g}$.

In particular, if Λ_1 and Λ_2 are two constant sections then their bracket is a constant section given by the Lie bracket on $\mathfrak{g}$. Note that the last two terms in Eq. (2.35) are due to the Leibniz rule. We will denote the transformation algebroid of an action of $\mathfrak{g}$ on M by $\mathfrak{g} \ltimes M$.

Example 2.3.11. Let (M, ω) be a pair where M is a smooth manifold and $\omega \in \Omega^2(M)$ is a closed $2-$form on M. Consider the vector bundle $A = TM \oplus (M \times \mathbb{R}) \to M$. Then, we may define the map

$$\begin{array}{rccc} \sharp : & A & \to & TM \\ & u_x \oplus (x,t) & \mapsto & u_x \end{array}.$$

In addition, note that the space $\Gamma(A)$ can be identified with the space

$$\overline{\Gamma(A)} := \{X \oplus f : X \in \mathfrak{X}(M), f \in \mathcal{C}^{\infty}(M)\}.$$

So, we constuct a bracket on $\Gamma(A)$ characterized by

$$[X \oplus f, Y \oplus g] = [X, Y] \oplus (X(g) - Y(f) + \omega(X, Y)),$$

for all $X \oplus f, Y \oplus g \in \overline{\Gamma(A)}$. These maps define a Lie algebroid structure on A which is transitive. In fact, the Jacobi identity is equivalent to the fact that ω is closed.

Example 2.3.12. Let $\tau : P \to M$ be a principal bundle with structure group G. Denote by $\phi : G \times P \to P$ the action of G on P. Now, suppose that $(A \to P, \sharp, [\cdot, \cdot])$ is a Lie algebroid, with vector bundle projection $\pi : A \to P$ and that $\overline{\phi} : G \times A \to A$ is an action of G on A such that π is a vector bundle action under the action $\overline{\phi}$ where for each $g \in G$, the pair $\left(\overline{\phi}_g, \phi_g\right)$ satisfies that

(1) $\sharp \circ \overline{\phi}_g = T\phi_g \circ \sharp$.

(2) $\left[\overline{\phi}_g \circ \Lambda_1 \circ \phi_g^{-1}, \overline{\phi}_g \circ \Lambda_2 \circ \phi_g^{-1}\right] = \overline{\phi}_g \circ [\Lambda_1, \Lambda_2] \circ \phi_g^{-1}, \ \forall \Lambda_1, \Lambda_2 \in \Gamma(A)$.

This fact will be equivalent to the fact that $\left(\overline{\phi}_g, \phi_g\right)$ is a Lie algebroid isomorphism. Let $\overline{\pi} : A/G \to M$ be the quotient vector bundle of π by the action of G. Then, we are going to construct a Lie algebroid structure on $\overline{\pi}$.

Denote by $\overline{\tau} : A \to A/G$ the quotient projection. Then, we may define the anchor map $\overline{\sharp} : A/G \to TM$ by

$$\overline{\sharp}(u) = T_{\pi(a)}\tau(\sharp(a)),$$

for all $u \in A/G$ and $a \in A$, where $\overline{\tau}(a) = u$.

Let $a, b \in A$ such that $\overline{\tau}(a) = \overline{\tau}(b) = u$. Then, there exists $g \in G$ such that

$$\overline{\phi}_g(b) = a.$$

Thus, since $\sharp \circ \overline{\phi}_g = T\phi_g \circ \sharp$, we have

$$\sharp(a) = \sharp\left(\overline{\phi}_g(b)\right) = T_{\tau(b)}\phi_g(\sharp(b)),$$

and therefore

$$T_{\pi(a)}\tau(\sharp(a)) = \{T_{\pi(b)}(\tau \circ \phi_g)\}(\sharp(b)) = T_{\pi(b)}\tau(\sharp(b)),$$

i.e., $\overline{\sharp}$ is well defined.

Furthermore, by construction

$$\overline{\sharp} \circ \overline{\tau} = T\tau \circ \sharp.$$

So, using that $\overline{\tau}$ is a submersion, the anchor is a smooth map. Finally, it is trivial that $\overline{\sharp}$ is a vector bundle morphism.

On the other hand, for each $\Lambda_1, \Lambda_2 \in \Gamma(A)^G$ and for each $g \in G$

$$\overline{\phi}_g \circ \Lambda_1 \circ \phi_{g^{-1}} = \Lambda_1, \qquad \overline{\phi}_g \circ \Lambda_2 \circ \phi_{g^{-1}} = \Lambda_2.$$

Then,

$$[\Lambda_1, \Lambda_2] = \left[\overline{\phi}_g \circ \Lambda_1 \circ \phi_{g^{-1}}, \overline{\phi}_g \circ \Lambda_2 \circ \phi_{g^{-1}}\right].$$

Using (2), we have

$$[\Lambda_1, \Lambda_2] = \overline{\phi}_g \circ [\Lambda_1, \Lambda_2] \circ \phi_{g^{-1}},$$

i.e. $[\Lambda_1, \Lambda_2] \in \Gamma(A)^G$. As a consequence, the Lie bracket on $\Gamma(A)$ restricts to $\Gamma(A)^G \cong \Gamma(A/G)$ and then, this structure induces a Lie algebra structure on $\Gamma(A/G)$. Finally, it is easy to prove that the Leibniz identity is satisfied. This kind of Lie algebroids are called *quotient Lie algebroids by the action of a Lie group.*

A particular but interesting example of this construction is obtained when we consider the tangent lift of a free and proper action of a Lie group on a manifold.

Example 2.3.13. Let $\pi : P \to M$ be a principal bundle with structure group G. Denote by ϕ the (left) action of G on P. Let $(TP \to P, Id_{TP}, [\cdot,\cdot])$ be the tangent algebroid and $\phi^T : G \times TP \to TP$ be the tangent lift of ϕ.

Then, ϕ^T satisfies the conditions of example 2.3.12. Thus, one may consider the quotient Lie algebroid $\left(TP/G \to M, \overline{\sharp}, \overline{[\cdot,\cdot]}\right)$ by the action of G. This algebroid is called the *Atiyah algebroid associated with the principal bundle* $\pi : P \to M$.

Note that, as we have seen, the space of sections can be considered as the space of invariant vector field by the action ϕ over M.

Next, we introduce the definition of a morphism in the category of Lie algebroids. However, the case of Lie algebroids is not as easy as the case of groupoids. The difficulty lies on the fact that, in general, a morphism between vector bundles does not induce a map between the modules of sections. This implies that a relation between the brackets of the space of sections from a morphisms of vector bundles is not immediately clear.
The definition of morphism of Lie algebroids was introduced by Pradines in [80]. Nevertheless, this definition was not simple enough to be used. Following the article of Pradines, Almeida and Kumpera gave another, more conceptual, definition in [2]. Even in that case, the difficulties do not

disappear at all and it is still difficult to work with it. It was necessary another definition (obtained from an observation made by Weinstein to Mackenzie about the Lie algebroid of an action groupoid) to solve this.
We will show a direct definition in terms of (Φ, ϕ) −decompositons of sections which is easy to understand.

Definition 2.3.14. *Let $\pi : A \to M$ and $\pi' : A' \to M'$ be vector bundles and (Φ, ϕ), with $\Phi : A' \to A$ and $\phi : M' \to M$ a vector bundle morphism. If $\Lambda \in \Gamma(A)$ and $\sigma \in \Gamma(A')$ satisfy*

$$\Phi \circ \sigma = \Lambda \circ \phi,$$

then we say that σ and Λ are (Φ, ϕ) −related and we write $\sigma \sim_{(\Phi,\phi)} \Lambda$. We also say that $\sigma \in \Gamma(A')$ is (Φ, ϕ) −projectable if it is (Φ, ϕ) −related to some $\Lambda \in \Gamma(A)$.

It is easy to prove that this relation is $\mathcal{C}^\infty(M)$-linear in the sense that if $\sigma \sim_{(\Phi,\phi)} \Lambda$, $\sigma' \sim_{(\Phi,\phi)} \Lambda'$ and $f \in \mathcal{C}^\infty(M)$, then

$$\sigma + \sigma' \sim_{(\Phi,\phi)} \Lambda + \Lambda'.$$

$$(f \circ \phi)\, \sigma \sim_{(\Phi,\phi)} f\Lambda.$$

In this way, projectable sections have a natural $\mathcal{C}^\infty(M)$ −module structure. However, we need a more general relationship which involves linearity over $\mathcal{C}^\infty(M')$.
The map ϕ determines an algebra morphism

$$\phi^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty\left(M'\right),$$

given by

$$\phi^*(f) = f \circ \phi, \ \forall f \in \mathcal{C}^\infty(M).$$

Then, ϕ^* provides a structure of $\mathcal{C}^\infty(M)$ −module to the space $\mathcal{C}^\infty(M') \times \Gamma(A)$. In this way, we can consider $\mathcal{C}^\infty(M') \otimes \Gamma(A)$, where the tensor product is over $\mathcal{C}^\infty(M)$.

Lemma 2.3.15. *Let $\phi^*\pi : \phi^*A \to M'$ be the pullback bundle. Then, $\Gamma(\phi^*A)$ is isomorphic, as a $\mathcal{C}^\infty(M')$ −module, to $\mathcal{C}^\infty(M') \otimes \Gamma(A)$. The isomorphism $F : \mathcal{C}^\infty(M') \otimes \Gamma(A) \to \Gamma(\phi^*A)$ is characterized by*

$$f' \otimes \Lambda \mapsto f'\overline{\Lambda},$$

where, $\overline{\Lambda} \in \Gamma(\phi^*A)$ is given by

$$\overline{\Lambda}(x) = (x, \Lambda(\phi(x))), \ \forall x \in M'.$$

Thus, with $\Phi^* : \Gamma(A') \to \Gamma(\phi^*A)$, for each $\Lambda' \in \Gamma(A')$, we can write

$$\Phi^*\left(\Lambda'\right) = \sum_{i=1}^{k} F\left(f_i' \otimes \Lambda_i\right), \tag{2.36}$$

for suitable $f_i' \in \mathcal{C}^\infty(M')$ and $\Lambda_i \in \Gamma(A)$, but such a representation does not need to be unique.
If we identify $\Gamma(\phi^*A)$ with the module $\Gamma_\phi(A)$ of smooth maps $f : M' \to A$ such that

$$\pi \circ f = \phi.$$

Then (2.36) becomes

$$\Phi \circ \Lambda' = \sum_{i=1}^{k} f_i'(\Lambda_i \circ \phi). \tag{2.37}$$

We refer to relation (2.37) as a (Φ, ϕ) −*decompositon* of Λ'. Note that the statement Λ' is (Φ, ϕ) −related to Λ is equivalent to

$$\Phi^*\left(\Lambda'\right) = F(1 \otimes \Lambda).$$

Thus, we are ready to give the definiton of Lie algebroid morphism.

Definition 2.3.16. Let $(A \to M, \sharp, [\cdot,\cdot])$, $\left(A' \to M', \sharp', [\cdot,\cdot]'\right)$ be Lie algebroids. A *morphism of Lie algebroids* is a vector bundle morphism $\Phi : A' \to A$, $\phi : M' \to M$ such that

$$\sharp \circ \Phi = T\phi \circ \sharp', \tag{2.38}$$

and such that for arbitrary $\Lambda'_1, \Lambda'_2 \in \Gamma(A')$ with (Φ, ϕ) −decompositions

$$\Phi \circ \Lambda'_1 = f_i \left(\Lambda^1_i \circ \phi\right), \Phi \circ \Lambda'_2 = g_j \left(\Lambda^2_j \circ \phi\right),$$

we have

$$\Phi \circ \left[\Lambda'_1, \Lambda'_2\right] = f_i g_j \left(\left[\Lambda^1_i, \Lambda^2_j\right] \circ \phi\right) + \tag{2.39}$$
$$+ {\Lambda'_1}^{\sharp'} (g_j) \left(\Lambda^2_j \circ \phi\right) - {\Lambda'_2}^{\sharp'} (f_i) \left(\Lambda^1_i \circ \phi\right).$$

In fact, the right-hand side of Eq. (2.39) is independent of the choice of the (Φ, ϕ) −decompositions of Λ'_1 and Λ'_2.

Now consider two morphisms of Lie algebroids, $\Phi' : A'' \to A'$, $\phi' : M'' \to M'$ and $\Phi : A' \to A$, $\phi : M' \to M$. One can observe that a (Φ', ϕ') −decompositon,

$$\Phi' \circ \Lambda''_1 = f''_i \left({\Lambda_1}'_i \circ \phi'\right),$$

combines with a (Φ, ϕ) −decompositon of each ${\Lambda_1}'_i$ to give a $(\Phi \circ \Phi', \phi \circ \phi')$ −decompositon, and verifies (2.39) for decompositons so formed. Therefore, checking that the condition for the anchor is satisfied, we have a category of Lie algebroids. We will denote this category by $\mathcal{LA}$.

Remark 2.3.17. In particular, if $\Lambda'_1 \sim_{(\Phi,\phi)} \Lambda_1$ and $\Lambda'_2 \sim_{(\Phi,\phi)} \Lambda_2$, then Eq. (2.39) reduces to

$$\Phi \circ \left[\Lambda'_1, \Lambda'_2\right] = \left[\Lambda_1, \Lambda_2\right] \circ \phi.$$

On the other hand, if $M = M'$ and $\phi = Id_M$ then Eq. (2.39) reduces to

$$\Phi \circ \left[\Lambda'_1, \Lambda'_2\right] = \left[\Phi \circ \Lambda'_1, \Phi \circ \Lambda'_2\right], \ \forall \Lambda'_1, \Lambda'_2 \in \Gamma\left(A'\right).$$

◇

Next, we are going to introduce the notion of Lie subalgebroid.

Definition 2.3.18. Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid. Suppose that A' is an embedded submanifold of A and M' is a inmersed submanifold of M with inclusion maps $i_{A'} : A' \hookrightarrow A$ and $i_{M'} : M' \hookrightarrow M$. A' is called a *Lie subalgebroid of* A if A' is a Lie algebroid on M' which is a vector subbundle of $\pi_{|M'}$, where $\pi : A \to M$ is the projection map of A, equipped with a Lie algebroid structure such that the inclusion is a morphism of Lie algebroids. A *reduced subalgebroid of* A is a transitive Lie subalgebroid with M as the base manifold.

Remark 2.3.19. Suppose that $M' \subseteq M$ is a closed submanifold then, using the $(i_{A'}, i_{M'})-$ decomposition and extending functions, it satisfies that for all $\Lambda'_1 \in \Gamma(A')$ there exists $\Lambda_1 \in \Gamma(A)$ such that

$$i_{A'} \circ \Lambda'_1 = \Lambda_1 \circ i_{M'}.$$

So, Eq. (2.39) reduces to

$$i_{A'} \circ \left[\Lambda'_1, \Lambda'_2\right]_{M'} = \left[\Lambda_1, \Lambda_2\right]_M \circ i_{M'}, \ \forall \Lambda'_1, \Lambda'_2 \in \Gamma\left(A'\right).$$

◇

Example 2.3.20. Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid over M. Then from lemma 2.3.2 and Remark 2.3.17 we deduce the anchor map $\sharp : A \to TM$ is a Lie algebroid morphism from A to the tangent algebroid of M.

Example 2.3.21. Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid over M and z a point of M. Then the inclusion map from the isotropy algebra $Ker(\sharp_z)$ of z to A is a Lie algebroid morphism.

Example 2.3.22. Let $\phi : M_1 \to M_2$ be a smooth map. Then $(T\phi, \phi)$ is a Lie algebroid morphism between the tangent algebroids TM_1 and TM_2.

Example 2.3.23. Let $\tau : P \to M$ be a principal bundle with structure group G and $(A \to P, \sharp, [\cdot,\cdot])$ be a Lie algebroid (with vector bundle projection $\pi : A \to P$) in the conditions of example 2.3.12. If $\overline{\pi} : A/G \to M$ is the quotient Lie algebroid by the action of the Lie group G then $(\overline{\tau}, \tau)$ is a Lie algebroid morphism. Remember that $\overline{\tau}$ is the quotient projection $\overline{\tau} : A \to A/G$.

Construction of the associated Lie algebroid

Now, it is time to justify the name of *infinitesimal groupoid* which was initially given to Lie algebroids. In order to do this, we will generalize the construction of the Lie algebra of a Lie group. As an important case of this construction we find the *1-jets algebroid*.
The process of contruction of the associated Lie algebroid to a Lie groupoid was formally extended for the case of subgroupoids (not necessarily Lie subgroupoids) of Lie groupoids in the article [51]. This is one of the papers included in the development of the thesis and it will be properly explained in section 4.1.
As a first step, we should generalize the notion of left-invariant vector fields of a Lie group.

Definition 2.3.24. Let $\Gamma \rightrightarrows M$ be a Lie groupoid with target map β. A vector field $\Theta \in \mathfrak{X}(\Gamma)$ is called *left-invariant* if it satisfies the following two properties:

(a) Θ is tangent to the $\beta-$fibres Γ^x, for all $x \in M$.

(b) For each $g \in \Gamma$, the left translation L_g preserves Θ.

Denote the space of smooth left-invariant vector fields on Γ by $\mathfrak{X}_L(\Gamma)$.

Similarly to the case of Lie groups, it is clear that the Lie bracket of two left-invariant vector fields is again a left-invariant vector field, say

$$[\mathfrak{X}_L(\Gamma), \mathfrak{X}_L(\Gamma)] \subset \mathfrak{X}_L(\Gamma). \tag{2.40}$$

On the other hand, $T\beta$ has constant rank. Thus, we may define the vector subbundle of $T\Gamma$ given by

$$\sqcup_{x\in M} T\Gamma^x = \sqcup_{g\in\Gamma} Ker(T_g\beta) = Ker(T\beta).$$

Let $\epsilon : M \to \Gamma$ be the section of identities. We define the pullback vector bundle on M,

$$\epsilon^*(Ker(T\beta)) = M \times_{\epsilon,\pi_\Gamma} Ker(T\beta), \tag{2.41}$$

where $\pi_\Gamma : T\Gamma \to \Gamma$ is the tangent bundle projection on Γ and $M \times_{\epsilon,\pi_\Gamma} Ker(T\beta)$ is the pullback space according to the following diagram

$$\begin{array}{ccc} M \times_{\epsilon,\pi_\Gamma} Ker\left(T\beta\right) & \xrightarrow{\quad pr_2 \quad} & Ker\left(T\beta\right) \\ \Big\downarrow {\scriptstyle pr_1} & & \Big\downarrow {\scriptstyle \pi_\Gamma} \\ M & \xrightarrow{\quad \epsilon \quad} & \Gamma \end{array}$$

where pr_i is the projection on the $i-$component of $M \times_{\epsilon,\pi_\Gamma} Ker\left(T\beta\right)$. Notice that $M \times_{\epsilon,\pi_\Gamma} Ker\left(T\beta\right)$ can be depicted as the disjoint union,

$$\sqcup_{x \in M} Ker\left(T_{\epsilon(x)}\beta\right). \tag{2.42}$$

We will denote this disjoint union by $A\Gamma$ and the projection will be denoted by $\pi^\epsilon : A\Gamma \to M$. Note that the sections of $A\Gamma$ are determined by smooth maps $\Lambda : M \to T\Gamma$ such that

(i) $T\beta \circ \Lambda = 0$

(ii) $\pi_\Gamma \circ \Lambda = \epsilon$

Thus, for each map $\Lambda \in \Gamma\left(A\Gamma\right)$ we can define the left-invariant vector field on Γ given by

$$\Theta^\Lambda\left(g\right) = T_{\epsilon(\alpha(g))}L_g\left(\Lambda\left(\alpha\left(g\right)\right)\right), \ \forall g \in \Gamma,$$

i.e., Θ^Λ is determined by the following equality

$$\Theta^\Lambda\left(\epsilon\left(x\right)\right) = \Lambda\left(x\right), \ \forall x \in M.$$

Conversely, if $\Theta \in \mathfrak{X}_L\left(\Gamma\right)$, then $\Lambda^\Theta = \Theta \circ \epsilon : M \to T\Gamma$ induces a section of $A\Gamma$ and, indeed, the correspondence $\Lambda \mapsto \Theta^\Lambda$ generates a linear isomorphism from $\Gamma\left(A\Gamma\right)$ to $\mathfrak{X}_L\left(\Gamma\right)$. With this identification $\Gamma\left(A\Gamma\right)$ inherits a Lie bracket from $\mathfrak{X}_L\left(\Gamma\right)$.

This construction is a natural extension of the Lie structure in the associated Lie algebra of a Lie group. In that case, we fix a Lie group G

and an element ξ of T_eG. Then, we constructed the associated left-invariant vector field by the equality

$$\Theta^{\xi}(e) = \xi.$$

Using this equality T_eG is endowed with a Lie algebra structure.

Finally, an anchor map can be defined as follows: identify $\mathcal{C}^{\infty}(M)$ with the space $\mathcal{C}^{\infty}_L(\Gamma)$ of left-invariant functions on Γ using the map given by $\Phi : f \in \mathcal{C}^{\infty}(M) \mapsto f \circ \alpha \in \mathcal{C}^{\infty}_L(\Gamma)$ ($f \circ \alpha \in \mathcal{C}^{\infty}_L(\Gamma)$ because $\alpha(g \cdot h) = \alpha(h)$, for all $(g,h) \in \Gamma_{(2)}$) with inverse map $\Phi^{-1} : f \in \mathcal{C}^{\infty}_L(\Gamma) \mapsto f \circ \epsilon \in \mathcal{C}^{\infty}(M)$.

Furthermore, like in the case of Lie groups, $\Theta \in \mathfrak{X}_L(\Gamma)$ if, and only if,

$$\Theta(f \circ L_g) = \Theta(f) \circ L_g, \ \forall g \in \Gamma, \ \forall f \in \mathcal{C}^{\infty}(\Gamma).$$

So, if $\Theta \in \mathfrak{X}_L(\Gamma)$ and $f \in \mathcal{C}^{\infty}_L(\Gamma)$, then we have

$$\Theta(f) \in \mathcal{C}^{\infty}_L(\Gamma).$$

In this way, we will define the anchor map as follows: let Λ be a section of $\Gamma(A\Gamma)$; then for each $f \in \mathcal{C}^{\infty}(M)$ we define

$$\Lambda^{\sharp}(f) = \Theta^{\Lambda}(f \circ \alpha) \circ \epsilon.$$

Thus, $\Lambda^{\sharp}(f) \in \mathcal{C}^{\infty}(M)$ for all $f \in \mathcal{C}^{\infty}(M)$. Furthermore, its inherits the Leibniz rule from Θ^{Λ} and so, $\sharp$ is well-defined.
Notice that, for each $x \in M$ and $f \in \mathcal{C}^{\infty}(M)$

$$\begin{aligned} \{\Lambda^{\sharp}(f)\}(x) &= \{\Theta^{\Lambda}(f \circ \alpha)\}(\epsilon(x)) \\ &= \{T_{\epsilon(x)}\alpha\left(\Theta^{\Lambda}(\epsilon(x))\right)\}(f) \\ &= \{T_{\epsilon(x)}\alpha(\Lambda(x))\}(f) \end{aligned}$$

i.e., it satisfies that

$$\sharp(\Lambda(x)) = T_{\epsilon(x)}\alpha\left(\Theta^{\Lambda}(\epsilon(x))\right) = T_{\epsilon(x)}\alpha(\Lambda(x)), \tag{2.43}$$

for all $\Lambda \in \Gamma(A\Gamma)$ and $x \in M$. Hence,

$$\sharp = \{T\alpha\}_{|A\Gamma}. \tag{2.44}$$

Therefore, $\sharp$ is a vector bundle morphism and it satisfies the Leibniz rule. So, $(A\Gamma \to M, \sharp, [\cdot,\cdot])$ is a Lie algebroid, called the *Lie algebroid associated to the Lie groupoid* $\Gamma \rightrightarrows M$, and denoted by $A\Gamma$.

Remark 2.3.25. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. For any $x \in M$, the associated Lie algebra to the isotropy Lie group Γ_x^x, $A(\Gamma_x^x)$ is isomorphic to the isotropy Lie algebra through x, i.e.,

$$A(\Gamma_x^x) \cong Ker(\sharp_x). \tag{2.45}$$

◇

Now, we can prove a result which shows the real nature of the given relation between Lie groupoids and Lie algebroid. This result will be proved for (not necessarily Lie) subgroupoids of a given Lie groupoid in section 4.1.

Theorem 2.3.26. *There is a natural functor A from the category of Lie groupoids to the category of Lie algebroids.*

Proof. We already have given the definition of the correspondence between objects ($\Gamma \rightrightarrows M \ \to A\Gamma$) and we will obtain the correspondence between morphisms.

Let $(\Phi, \phi) : \Gamma_1 \rightrightarrows M_1 \to \Gamma_2 \rightrightarrows M_2$ be a Lie groupoid morphism, with $\Phi : \Gamma_1 \to \Gamma_2$ and $\phi_1 : M_1 \to M_2$. Then, (Φ, ϕ) induces a morphism of Lie algebroids from $A\Gamma_1$ to $A\Gamma_2$ given by (Φ_*, ϕ) where

$$\Phi_* = T\Phi_{|A\Gamma_1} \tag{2.46}$$

So, if $v_{\epsilon_1(x)} \in Ker\left(T_{\epsilon_1(x)}\beta_1\right)$ then

$$\Phi_*\left(v_{\epsilon_1(x)}\right) = T_{\epsilon_1(x)}\Phi_x\left(v_{\epsilon_1(x)}\right),$$

where $\Phi_x : \beta_1^{-1}(x) \to \beta_2^{-1}(\phi(x))$, for each $x \in M$. Now,

$$\pi_{\Gamma_2}\left(T_{\epsilon_1(x)}\Phi\left(v_{\epsilon_1(x)}\right)\right) = \Phi(\epsilon_1(x)),$$

and using that (Φ, ϕ) is a morphism of Lie groupoids, it follows

$$\pi_{\Gamma_2}\left(T_{\epsilon_1(x)}\Phi\left(v_{\epsilon_1(x)}\right)\right) = \epsilon_2(\phi(x)).$$

Furthermore, using again that (Φ, ϕ) is a morphism of Lie groupoids,

$$\begin{array}{rcl} T_{\Phi(\epsilon_1(x))}\beta_2\left(T_{\epsilon_1(x)}\Phi\left(v_{\epsilon_1(x)}\right)\right) & = & T_{\epsilon_1(x)}\left(\beta_2 \circ \Phi\right)\left(v_{\epsilon_1(x)}\right) \\ = T_{\epsilon_1(x)}\left(\phi \circ \beta_1\right)\left(v_{\epsilon_1(x)}\right) & = & 0. \end{array}$$

Thus, $\Phi_*\left(v_{\epsilon_1(x)}\right) \in A\Gamma_2$, i.e.,

$$\Phi_* : A\Gamma_1 \to A\Gamma_2.$$

Also, it is trivial to show that the following diagram is commutative

$$\begin{array}{ccc} A\Gamma_1 & \xrightarrow{\pi^{\epsilon_1}} & M_1 \\ \downarrow \Phi_* & & \downarrow \phi \\ A\Gamma_2 & \xrightarrow{\pi^{\epsilon_2}} & M_2 \end{array}$$

Additionally, for all $x \in M_1$ the map

$$\Phi_{*|A\Gamma_{1x}} : A\Gamma_{1x} \to A\Gamma_{2x},$$

is linear so that we have that the map (Φ_*, ϕ) is a vector bundle morphism. Finally, we must study how (Φ_*, ϕ) works with the anchor map and the bracket of sections:

a) Observe that, for each $i = 1, 2$ we have

$$\sharp_i\left(\Lambda\left(x\right)\right) = T_{\epsilon_i(x)}\alpha_i\left(\Lambda\left(x\right)\right),$$

for any $\Lambda \in \Gamma\left(A\Gamma_i\right)$ and $x \in M_i$. Using this identity it is trivial that the following diagram

$$\begin{array}{ccc} A\Gamma_1 & \xrightarrow{\Phi_*} & A\Gamma_2 \\ \sharp_1 \big\downarrow & & \big\downarrow \sharp_2 \\ TM_1 & \xrightarrow{T\phi} & TM_2 \end{array}$$

is a commutative diagram, i.e., $\sharp_2 \circ \Phi_* = T\phi \circ \sharp_1$.

b) Let Λ be a section of $A\Gamma_1$ with (Φ_*, ϕ) −decompositon,

$$\Phi_* \Lambda = f_i \left(\Lambda_i \circ \phi \right). \tag{2.47}$$

Then, for all $g \in \Gamma_1$,

$$\begin{aligned} \left\{ T\Phi \circ \Theta^{\Lambda} \right\}(g) &= T_g \Phi \left(T_{\epsilon_1(\alpha_1(g))} L_g \left(\Lambda \left(\alpha_1 \left(g \right) \right) \right) \right) \\ &= T_{\epsilon_1(\alpha_1(g))} \left(\Phi \circ L_g \right) \left(\Lambda \left(\alpha_1 \left(g \right) \right) \right). \end{aligned}$$

Since (Φ, ϕ) is a morphism of Lie groupoids

$$\Phi \circ L_g = L_{\Phi(g)} \circ \Phi.$$

Then,

$$\begin{aligned} & \left\{ T\Phi \circ \Theta^{\Lambda} \right\}(g) = \\ =\ & T_{\epsilon_1(\alpha_1(g))} \left(L_{\Phi(g)} \circ \Phi \right) \left(\Lambda \left(\alpha_1 \left(g \right) \right) \right) \\ =\ & T_{\Phi(\epsilon_1(\alpha_1(g)))} L_{\Phi(g)} \left(T_{\epsilon_1(\alpha_1(g))} \Phi \left(\Lambda \left(\alpha_1 \left(g \right) \right) \right) \right) \\ =\ & T_{\Phi(\epsilon_1(\alpha_1(g)))} L_{\Phi(g)} \left\{ \left(T\Phi \circ \Lambda \right) \left(\alpha_1 \left(g \right) \right) \right\} \\ =\ & \left(f_i \circ \alpha_1 \left(g \right) \right) \left\{ T_{\Phi(\epsilon_1(\alpha_1(g)))} L_{\Phi(g)} \right\} \left(\Lambda_i \circ \phi \circ \alpha_1 \left(g \right) \right) \\ =\ & \left(f_i \circ \alpha_1 \left(g \right) \right) \left\{ T_{\Phi(\epsilon_1(\alpha_1(g)))} L_{\Phi(g)} \right\} \left(\Lambda_i \circ \alpha_2 \circ \Phi \left(g \right) \right) \\ =\ & \left(f_i \circ \alpha_1 \left(g \right) \right) \left(\Theta^{\Lambda_i} \left(\Phi \left(g \right) \right) \right). \end{aligned}$$

Thus, we have got the following identity

$$T\Phi \circ \Theta^{\Lambda} = (f_i \circ \alpha_1) \left(\Theta^{\Lambda_i} \circ \Phi\right).$$

Finally, using this identity and that $(T\Phi, \Phi)$ is a Lie algebroid morphism between the tangent algebroids, it is a routinary exercise to prove the identity (2.39).

□

The morphism induced by a morphism (Φ, ϕ) of Lie groupoids over the associated Lie algebroids will be denoted by $A\Phi$.

Now, we are going to give some examples of the above general construction.

Example 2.3.27. Let M be a smooth manifold and $M \times M \rightrightarrows M$ be the pair groupoid (see example 2.2.6). Then, the vector bundle $\epsilon^* (Ker\,(t\beta))$ can be seen as the tangent bundle $\pi_M : TM \to M$. With this, it follows that the associated Lie algebroid to $M \times M \rightrightarrows M$ is the tangent algebroid.

Example 2.3.28. Let M be a manifold and G be a Lie group. Consider the trivial Lie groupoid on M with group G (see example 2.2.18). Then, the associated Lie algebroid is the trivial Lie algebroid on M with structure algebra $\mathfrak{g}$ (see example 2.3.8), i.e., $TM \oplus (M \times \mathfrak{g}) \to M$.

Example 2.3.29. Let $\pi : P \to M$ be a principal bundle with structure group G. Denote by $\phi : G \times P \to P$ the action of G on P.

Now, suppose that $\Gamma \rightrightarrows P$ is a Lie groupoid, with $\overline{\phi} : G \times \Gamma \to \Gamma$ a free and proper action of G on Γ such that, for each $g \in G$, the pair $\left(\overline{\phi}_g, \phi_g\right)$ is an isomorphism of Lie groupoids. So, we may construct the quotient Lie groupoid by the action of a Lie group, $\Gamma/G \rightrightarrows M$ (see example 2.2.20).

Then, by construction, we may identify $A\,(\Gamma/G)$ with the quotient Lie algebroids by the action of a Lie group, $A\Gamma/G$ (see example 2.3.12).

As a particular case, we may give the following interesting example.

Example 2.3.30. Let $\pi : P \to M$ be a principal bundle with structure group G and $Gauge\,(P)$ be the Gauge groupoid (see example 2.2.21). Then, the associated Lie algebroid to $Gauge\,(P)$ is the Atiyah algebroid associated with the principal bundle $\pi : P \to M$ (see example 2.3.13).

Example 2.3.31. Let $\Phi(A) \rightrightarrows M$ be the frame groupoid. Then $A\Phi(A)$ is called *frame algebroid* (see example 2.2.22). As a particular case, $A\Pi^1(M,M)$ is called $1-$*jets algebroid*.

Let (x^i) be a local coordinate system defined on some open subset $U \subseteq M$, using Eq. (2.25) and Eq. (2.41) we can consider local coordinates on $A\Pi^1(M,M)$ as follows

$$A\Pi^1(U,U) : \left(\left(x^i, x^i, \delta^i_j\right), 0, v^i, v^i_j\right) \cong \left(x^i, v^i, v^i_j\right). \qquad (2.48)$$

We will pay an special attention to the $1-$jet algebroid because of the fundamental role which will play in section 3.1 [54].
Let us describe a particular but important family of section of $A\Pi^1(M,M)$. Consider a vector field Θ on M. Denote by $\varphi^\Theta_t : U_t \to U_{-t}$ the (local) flow of Θ. Then, for each t we can construct a diffeomorphism,

$$\Pi\varphi^\Theta_t : \Pi^1(U_{-t}, \mathcal{B}) \to \Pi^1(U_t, \mathcal{B}),$$

such that

$$\Pi\varphi^\Theta_t(g) = g \cdot j^1_{\varphi^\Theta_{-t}(\alpha(g)),\alpha(g)}\varphi^\Theta_t.$$

So, this flow induces a left-invariant vector field on $\Pi^1(M,M)$ which generates a section of $A\Pi^1(M,M)$ denoted by $j^1\Theta$. $j^1\Theta$ is called the *complete lift of* Θ *on* $\Pi^1(M,M)$.
Let (x^i) be a local chart of M and $\left(x^i, y^j, y^j_i\right)$ be the induced local chart of $\Pi^1(M,M)$. Assume that, locally, Θ is written as follows,

$$\Theta = \Theta^i \frac{\partial}{\partial x^i}.$$

Then, locally, $j^1\Theta$ is expressed in the following way:

$$j^1\Theta = -\Theta^i \frac{\partial}{\partial x^i} + \frac{\partial \Theta^j}{\partial x^i}\frac{\partial}{\partial y^j_i}. \qquad (2.49)$$

Notice that $j^1\Theta$ can be equivalently induced by a $1-$jet of Θ. Thus, $A\Pi^1(M,M)$ can be interpreted as the bundle of $1-$jets of vector fields on M.

Example 2.3.32. Let $\tilde{J}^1(FM)$ be the second-order non-holonomic groupoid over a manifold M (see example 2.2.23) and $\left(x^i\right)$ be a local coordinate system on an open set $U \subseteq M$. Using Eq. (2.29) we can construct induced local coordinates over $A\tilde{J}^1(FM)$ as follows:

$$A\tilde{J}^1(FU) : \left(\left(\left(x^i\right), \left(x^i, \delta^i_j\right), \delta^i_j, 0\right), v^i, v^i_j, 0, v^i_{,j}, v^i_{j,k}\right) \cong \tag{2.50}$$

$$\cong \left(x^i, v^i, v^i_j, v^i_{,j}, v^i_{j,k}\right).$$

$A\tilde{J}^1(FM)$ is called the *second-order non-holonomic algebroid over M*. We will denote the anchor of this Lie algebroid by $\overline{\sharp}$.
The second-order non-holonomic algebroid will be very important in section 3.2 [52].

Let us now give an specific shape for the 1−jets algebroid. The process shown here is naturally generalized for the frame algebroid in [64].
Let $Der(TM)$ be the collection of all derivations on M. Remember that (see Box 2.1) a derivation $D : \mathfrak{X}(M) \to \mathfrak{X}(M)$ on M is a $\mathbb{R}-$linear map with base vector field $\Theta \in \mathfrak{X}(M)$ such that for each $f \in \mathcal{C}^\infty(M)$ and $\Xi \in \mathfrak{X}(M)$,

$$D(f\Xi) = fD(\Xi) + \Theta(f)\,\Xi.$$

- A *zeroth-order differential operator on M* is a $\mathcal{C}^\infty(M)-$linear endomorphism $\mathfrak{X}(M) \to \mathfrak{X}(M)$.

- A *first-order differential operator on M* is a $\mathbb{R}-$linear map $D : \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that for each $f \in \mathcal{C}^\infty(M)$, the map

$$\begin{array}{ccc} \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ \Xi & \mapsto & D(f\Xi) - fD(\Xi). \end{array}$$

  is a zeroth-order differential operator on M. Equivalently, for all $f, g \in \mathcal{C}^\infty(M)$ and $\Xi \in \mathfrak{X}(M)$,

$$D(fg\Xi) = fD(g\Xi) + gD(f\Xi) - fgD(\Xi).$$

Notice that the space of zeroth-order differential operators is contained in the space of derivations on M $(X = 0)$ and this space is contained in the space of first-order differential operators.
Now, associated to any first-order differential operator, D, there is a map from $1-$forms on M to zeroth-order differential operators on M, called *symbol of* D, which is determined by

$$\{\sigma(D)(df)\}(\Xi) = [D, f](\Xi) = D(f\Xi) - fD(\Xi),$$

for all $f \in \mathcal{C}^{\infty}(M)$ and $\Xi \in \mathfrak{X}(M)$.
Thus, D is a derivation on M if, and only if, there exists a vector field Θ on M such that for all $\Lambda \in \Omega^1(M)$,

$$\sigma(D)(\Lambda) = \Lambda(\Theta)\, Id_{\mathfrak{X}(TM)}.$$

With this, the symbol of D evaluated at any $1-$form Λ at a point $x \in M$ is a scalar multiple of the identity map of the fibre T_xM over x; $\sigma(D)(\Lambda)(x) = \Lambda(x)(\Theta(x))\, Id_{T_xM}$. We have thus obtained that a first-order differential operator is a derivation if, and only if, it has scalar symbol. Furthermore, it is obvious that $\sigma(D) = 0$ if, and only if, D is a zeroth-order differential operator.
Now, the space of first-order differential operators on M can be considered as the space of sections of a vector bundle $Diff^1(M)$ on M. So, we can define σ as a vector bundle morphism

$$\sigma : Diff^1(M) \to Hom(T^*M, End(TM)),$$

which will be called the *symbol of* M.
It turns out that σ is a surjective submersion and its kernel the zeroth-order differential operators. Thus, σ induces a short exact sequence of vector bundles over M

$$End(TM) \hookrightarrow Diff^1(M) \to Hom(T^*M, End(TM)).$$

Next, we can define $\mathfrak{D}(TM)$ to be the pullback vector bundle defined by the symbol map and the injective map

$$\begin{array}{rccl} I: & TM & \to & Hom(T^*M, End(TM)) \\ & v_x & \mapsto & I(v_x), \end{array}$$

where for each $\Lambda_x \in T_x^* M$ and $w_x \in T_x M$

$$\{I(v_x)(\Lambda_x)\}(w_x) = \Lambda_x(v_x)\, w_x,$$

according to the diagram

$$\begin{array}{ccc}
\mathfrak{D}(TM) & \longrightarrow & TM \\
\big\downarrow & & \big\downarrow I \\
Diff^1(M) & \xrightarrow{\ \sigma\ } & Hom(T^*M, End(TM))
\end{array}$$

Furthermore, taking into account that the left-hand vertical arrow is an injective inmersion we can consider $\mathfrak{D}(TM)$ as a subbundle of $Diff^1(TM)$. We will denote the top arrow by a and, clearly, as we have noticed before, the kernel of a is $End(TM)$. So, using a, we can consider another exact sequence

$$End(TM) \hookrightarrow \mathfrak{D}(TM) \to TM,$$

where, taking into account the map I, the space of sections of $\mathfrak{D}(TM)$ is, indeed, identifiable with the space $Der(TM)$ of the derivations on M.
In fact, we can endow the vector bundle $\mathfrak{D}(TM)$ with a Lie algebroid structure.

- Let D_1, D_2 be derivations on M, we can define $[D_1, D_2]$ as the commutator, i.e.,

  $$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

  A simple computation shows that the commutator of two derivations is again a derivation, indeed, the base vector field of $[D_1, D_2]$ is given by

  $$[\Theta_1, \Theta_2], \tag{2.51}$$

  where Θ_1 and Θ_2 are the base vector fields of D_1 and D_2 respectively.

- Let D be a derivation on M, then $D^{\sharp}$ is its base vector field.

Thus, with this structure $\mathfrak{D}(TM)$ is a transitive Lie algebroid called the *Lie algebroid of derivations on M.*
Note that in this Lie algebroid the fibre-wise linear sections of $\sharp$ are $\mathcal{C}^{\infty}(M)-$linear maps from $\mathfrak{X}(M)$ to $Der(TM)$. So, the space of fibre-wise linear sections of $\sharp$ is, indeed, the space of covariant derivatives on M (see Box 2.1). In fact, it is easy to see that *a covariant derivative ∇ is a Lie algebroid morphism (from the tangent algebroid to the algebroid of derivations) if, and only if, ∇ is flat.*
Finally, it is turn to relate this algebroid with the $1-$jets Lie algebroid. Consider $\Lambda \in \Gamma\left(A\Pi^{1}(M,M)\right)$ and Θ^{Λ} its associated left-invariant vector field on $\Pi^{1}(M,M)$. Denote by $\varphi_{t}^{\Lambda}: \mathcal{U}_{t} \to \mathcal{U}_{-t}$ the flow of Θ^{Λ}.
Then, we can define a (local) linear map $\left(\varphi_{t}^{\Lambda}\right)^{*}: \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying

$$\{\left(\varphi_{t}^{\Lambda}\right)^{*}(\Theta)\}(x) = \varphi_{t}^{\Lambda}(\epsilon(x))\left(\Theta\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\right),$$

for each $\Theta \in \mathfrak{X}(M)$ and $x \in M$. Thus, we can define the following derivation on M,

$$D^{\Lambda} = \frac{\partial}{\partial t_{|0}}\left(\varphi_{t}^{\Lambda}\right)^{*}.$$

In other words, for any $\Theta \in \mathfrak{X}(M)$ and $x \in M$ we have

$$D^{\Lambda}\Theta(x) \quad = \quad \frac{\partial}{\partial t_{|0}}\left(\varphi_{t}^{\Lambda}(\epsilon(x))\left(\Theta\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\right)\right).$$

Notice that, for all $f \in \mathcal{C}^{\infty}(M)$

$$\begin{aligned}
& D^{\Lambda} f\Theta(x) = \\
= \quad & \frac{\partial}{\partial t_{|0}}\left(\varphi_{t}^{\Lambda}(\epsilon(x))\left(f\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\Theta\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\right)\right) \\
= \quad & \frac{\partial}{\partial t_{|0}}\left(f\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\varphi_{t}^{\Lambda}(\epsilon(x))\left(\Theta\left(\left(\alpha \circ \varphi_{t}^{\Lambda}\right)(\epsilon(x))\right)\right)\right) \\
= \quad & \Lambda^{\sharp}(x)(f)\Theta(x) + f(x)D^{\Lambda}\Theta(x).
\end{aligned}$$

It is immediate to prove that for each $\Theta \in \mathfrak{X}(M)$ one has that

$$D^{j^1\Theta}\Xi = [\Theta, \Xi], \ \forall \Xi \in \mathfrak{X}(M). \tag{2.52}$$

This construction gives us a linear map between the sections of the $1-$jets Lie algebroid and the algebroid of derivations which induces a Lie algebroid isomorphism $\mathcal{D} : A\Pi^1(M,M) \to \mathfrak{D}(TM)$ over the identity map on M.

In fact, let us show how the map $\mathcal{D}$ looks locally :

Lemma 2.3.33. *Let M be a manifold and Λ be a section of the $1-$jets algebroid with local expression*

$$\Lambda\left(x^i\right) = \left(x^i, \Lambda^j, \Lambda^j_i\right).$$

The matrix Λ^j_i is (locally) the associated matrix to D^Λ, i.e.,

$$D^\Lambda\left(\frac{\partial}{\partial x^i}\right) = \Lambda^j_i \frac{\partial}{\partial x^j},$$

and the base vector field of D^Λ is $\Lambda^\sharp$ which is given locally by $\left(x^i, \Lambda^j\right)$.

Proof. Let $\Lambda \in \Gamma\left(A\Pi^1(M,M)\right)$ be a section of the $1-$jets algebroid and Θ^Λ its associated left-invariant vector field over $\Pi^1(M,M)$. Considering the flow of Θ^Λ, $\{\varphi^\Lambda_t : \mathcal{U}_t \to \mathcal{U}_{-t}\}$ we have by left invariance that

$$\varphi^\Lambda_t(x) = \xi^{-1} \cdot \varphi^\Lambda_t(\xi), \ \forall x \in \alpha(\mathcal{U}_t),$$

where $\xi \in \mathcal{U}_t \cap \alpha^{-1}(x)$.
Now, let us take local coordinate systems $\left(x^i\right)$ and $\left(y^j\right)$ and its induced local coordinates over Λ, then

$$\Lambda\left(x^i\right) = \left(x^i, \Lambda^j, \Lambda^j_i\right).$$

Thus, the associated left-invariant vector field is (locally) as follows

$$\Theta^\Lambda\left(x^i, y^j, y^j_i\right) = \left(\left(x^i, y^j, y^j_i\right), \Lambda^j, 0, y^j_l \cdot \Lambda^l_i\right).$$

Therefore, its flow

$$\varphi_t^{\Lambda}\left(x^i, y^j, y_i^j\right) = \left(\psi_t^{\Lambda}\left(x^i\right), y^j, y_l^j \cdot \overline{\varphi}_t^{\Lambda}\left(x^i\right)\right),$$

satisfies that

(i) ψ_t is the flow of $\Lambda^{\sharp}$.

(ii) $\dfrac{\partial}{\partial t_{|t=0}}\left(\overline{\varphi}_t^{\Lambda}\left(x^i\right)\right) = \Lambda_i^j$.

Then,

$$\begin{aligned}\left(\varphi_t^{\Lambda}\right)^*\left(\frac{\partial}{\partial x^k_{|x^i}}\right) &= \left(\psi_t\left(x^i\right), x^i, \overline{\varphi}_t\left(x^i\right)\right)\left(\frac{\partial}{\partial x^k_{|\psi_t\left(x^i\right)}}\right)\\ &= \overline{\varphi}_t\left(x^i\right)\frac{\partial}{\partial x^k_{|x^i}}\end{aligned}$$

Hence,

$$D^{\Lambda}\frac{\partial}{\partial x^k} = \Lambda_k^j \frac{\partial}{\partial x^j},$$

i.e., the matrix Λ_i^j is (locally) the associated matrix to D^{Λ}. □

Notice that using this isomorphism, we can consider a one-to-one map from fibre-wise linear sections of $\sharp$ in $A\Pi^1(M,M)$ to covariant derivatives over M. Thus, having a fibre-wise linear section Δ of $\sharp$ in $A\Pi^1(M,M)$ we will denote its associated covariant derivative by ∇^{Δ}. Furthermore, Δ is a morphism of Lie algebroids if, and only if, ∇^{Δ} is flat.

Let Δ be a fibre-wise linear section of $\sharp$ in $A\Pi^1(M,M)$ and ∇^{Δ} be its associated covariant derivative. Thus, for each $\left(x^i\right)$ local coordinate system on M

$$\Delta\left(x^i, \frac{\partial}{\partial x^j}\right) = \left(x^i, \frac{\partial}{\partial x^j}, \Delta_i^j\right),$$

where Δ_i^j depends on $\dfrac{\partial}{\partial x^j}$ linearly. Thus, we will change the notation as follows

$$\Delta_i^j \left(x^l, \frac{\partial}{\partial x^k} \right) = \Delta_{i,k}^j \left(x^l \right). \tag{2.53}$$

Therefore,

$$\nabla^{\Delta}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = D^{\Delta\left(\frac{\partial}{\partial x^j}\right)} \frac{\partial}{\partial x^i} = \Delta_{i,j}^k \frac{\partial}{\partial x^k},$$

where $\Delta\left(\dfrac{\partial}{\partial x^j}\right)$ is the (local) section of $A\Pi^1(M,M)$ given by

$$\Delta\left(\frac{\partial}{\partial x^j}\right)(x) = \Delta(x)\left(\frac{\partial}{\partial x^j_{|x}}\right).$$

So, $\Delta_{i,j}^k$ are just the Christoffel symbols of ∇^{Δ}.

Integrability of Lie algebroids

In [80–82], Pradines exposed the possibility of working on a complete Lie theory for Lie groupoids and Lie algebroids presenting also new results.
As we mentioned, Pradines generalized the construction of the associated Lie algebra to a Lie group to the case of Lie groupoids in order to introduce the structure of Lie algebroid. In fact, this construction is a functor between these categories (see theorem 2.3.26). This functor was given by Pradines [80] and is detailed by Mackenzie in [63] for the case of Lie algebroids with the same base and by Higgins and Mackenzie in [48] for the case of Lie algebroids with differents bases.
This Lie functor, in the case of Lie groupoids, preserves several fundamental properties. A natural question is the following: Are the same properties preserved in the case of Lie groupoids and Lie algebroids? It turns out that the answer is negative. In fact, there is a Lie theory for these two kind of objects, presented by Pradines in [79–82], in a collection of notes where the proofs of many results are, in fact, omitted.
In this way, there is a need of extending the three Lie's fundamental

theorems (see [21]) for Lie groups and Lie algebras:

Lie's first fundamental theorem
Any integrable Lie algebra can be integrated to a simply connected Lie group.

Lie's second fundamental theorem
Any morphism between integrable Lie algebras can be integrated to a morphism of Lie groups.

Lie's third fundamental theorem
Any Lie algebra can be integrated to a Lie group.

Actually it has been proved that Lie's first fundamental theorem and Lie's second fundamental theorem can be extended to the context of Lie groupoids and Lie algebroids. In order to generalize Lie's third fundamental theorem, in [82] J. Pradines presents the next question: is any Lie algebroid integrable (see definition 2.3.34)? For a long time people thought that there were not non-integrable Lie algebroids, such as J. Pradines believed in [82]. Nevertheless, R. Almeida and P. Molino showed in [3] that this assumption did not hold and that there are not integrable Lie algebroids. In [17] authors give necessary and sufficient conditions for the integrability of any Lie algebroid.

Definition 2.3.34. A Lie algebroid $(A \to M, \sharp, [\cdot,\cdot])$ is called *integrable* if it is isomorphic to the Lie algebroid $A\Gamma$ associated to a Lie groupoid $\Gamma \rightrightarrows M$. If this is the case, then $\Gamma \rightrightarrows M$ is called an integral of $(A \to M, \sharp, [\cdot,\cdot])$.

Note that, if $\mathcal{U} \subseteq \Gamma$ is an open reduced Lie subgroupoid of $\Gamma \rightrightarrows M$, then it is clear that $A\mathcal{U}$ and $A\Gamma$ are isomorphic. From now on, we will assume that M is connected.

Definition 2.3.35. A Lie groupoid $\Gamma \rightrightarrows M$ is said to be *target-connected* if Γ^x is connected for any $x \in M$. It is said to be *target-simply connected* if each Γ^x is connected and simply connected.

Example 2.3.36. Let M be a connected smooth manifold and $\mathcal{F}$ be a regular foliation in M (see Appendix B). The *monodromy groupoid* $Mon\,(M, \mathcal{F})$, is a groupoid over M with the following properties:

(i) For each $x, y \in M$, the set of morphisms from x to y is given by

$$\begin{cases} \Pi_{\mathcal{F}(x)}\,(x,y) & if \quad y \in \mathcal{F}\,(x) \\ \emptyset & if \quad y \notin \mathcal{F}\,(x) \end{cases}$$

where $\mathcal{F}\,(x) \in \mathcal{F}$ is the leaf through x and $\Pi_{\mathcal{F}(x)}\,(x,y)$ is the set of homotopy classes (relative to end-points) of paths in $\mathcal{F}\,(x)$ from x to y.

(b) The multiplication is induced by the concatenation of paths.

In particular, the isotropy groups of the monodromy groupoid are the fundamental groups of the leaves and the orbits are the leaves of $\mathcal{F}$. If $\mathcal{F}$ consists of just one leaf, i.e., the connected manifold M itself, then the groupoid $Mon\,(M, \mathcal{F})$ is called the *fundamental groupoid* of M which is transitive (provided that M is connected), and its isotropy groups are isomorphic to the fundamental groups of M.
Let $\beta : Mon\,(M, \mathcal{F}) \to M$ be the target map of the monodromy groupoid, then

$$Mon\,(M, \mathcal{F})^x = \Pi_{\mathcal{F}(x)}\,(x)\,,$$

where $\Pi_{\mathcal{F}(x)}\,(x)$ is the set of path classes of paths in $\mathcal{F}\,(x)$ ending at x. Thus, using the proof of the theorem of the existence of the Universal Covering Space (see, for example, [62]), we get that $\Pi_{\mathcal{F}(x)}\,(x)$ is simply-connected and that the map $q : \Pi_{\mathcal{F}(x)}\,(x) \to \mathcal{F}\,(x)$ given by

$$q\,([\gamma]) = \gamma\,(0)\,,$$

is a covering projection. Hence, the monodromy groupoid is a target-simply connected Lie groupoid.

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define the set

$$\Gamma^o = \sqcup_{x \in M} \Gamma^{o\,x},$$

where Γ^{ox} is the connected component of Γ^{x} with $\epsilon(x) \in \Gamma^{ox}$.
If $g, h \in \Gamma^{o}$ with $g \in \Gamma^{ox}$ and $h \in \Gamma^{oy}$ then, by connexity, there exist $\gamma_g : I \to \Gamma^{ox}$ and $\gamma_h : I \to \Gamma^{oy}$ such that $\gamma_g(0) = g, \gamma_g(1) = \epsilon(x), \gamma_h(1) = h$ and $\gamma_h(0) = \epsilon(y)$. Furthermore, using that M is a connected manifold, there exists $\gamma : I \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Thus, the smooth path given by

$$\rho = \gamma_h * (\epsilon \circ \gamma) * \gamma_g : I \to \Gamma^{o},$$

being $*$ concatenation, satisfies that $\rho(0) = g$ and $\rho(1) = h$. So, Γ^{o} is a connected subset of Γ.
On the other hand, considering the vertical distribution for $\beta : \Gamma \to M$, $Ker(T\beta)$ given by

$$g \mapsto Ker(T_g\beta), \ \forall g \in \Gamma,$$

is an integral distribution of rank k whose leaves are the connected components of the $\beta-$fibres of Γ. Therefore, for each $x \in M$ we may consider a foliation chart of $\epsilon(x)$,

$$\phi : U \to \mathbb{R}^{n-k} \times \mathbb{R}^{k}.$$

By restricting, we assume that the following fact: If $\Gamma^{y} \cap U \neq \emptyset$, then $\epsilon(y) \in U$. Hence, by connexity, it is clear that $U \subseteq \Gamma^{o}$. Taking V the union of such U we obtain an open neighbourhood of $\epsilon(M)$ in Γ which is contained in Γ^{o}.
Now, Γ^{o} is the union of these leaves of the foliation which intersect the open neighbourhood and so is itself open. It follows the following result:

Lemma 2.3.37. *Γ^{o} is an open and connected Lie subgroupoid of $\Gamma \rightrightarrows M$ over M.*

Proof. Taking into account that Γ^{o} is an open subset of Γ, we only have to verify that the structure maps can be restricted to Γ^{o}. □

Therefore, any Lie groupoid has an open and target-connected Lie subgroupoid and then, the Lie algebroids associated are isomorphic.

Theorem 2.3.38 (Lie I). *Let $\Gamma \to M$ be a Lie groupoid. There exists a target-simply connected Lie groupoid $\overline{\Gamma} \rightrightarrows M$ and a morphism of Lie groupoids $\overline{F} : \overline{\Gamma} \to \Gamma$, inducing a Lie algebroid isomorphism $A\overline{\Gamma} \to A\Gamma$.*

Proof. Using lemma 2.3.37, we can assume that Γ is source-connected. Let $\mathcal{F}$ be the regular foliation of Γ given by the $\beta-$fibres, and let $Mon\,(\Gamma, \mathcal{F})$ be its monodromy groupoid over Γ (see example 2.3.36). Let us denote the target (resp. source) map of $Mon\,(\Gamma, \mathcal{F})$ by β_M (resp. α_M). Over the monodromy groupoid can be defined the following equivalence relation

$$[\gamma] \sim [\rho],$$

if, and only if, there exist $g \in \Gamma$ such that $[L_g \circ \gamma] = [\rho]$. Then, the qoutient space defines a Lie groupoid $\overline{\Gamma} = Mon\,(\Gamma, \mathcal{F})\,/\Gamma \rightrightarrows M$. Since any monodromy groupoid is target-simply connected we can check that $\overline{\Gamma} \rightrightarrows M$ is again target-simply connected. Finally, let us consider the smooth map

$$F : Mon\,(\Gamma, \mathcal{F}) \to \Gamma,$$

given by $F([\gamma]) = \gamma(1) \cdot \gamma(0)^{-1}$ on arrows of $Mon\,(\Gamma, \mathcal{F})$. Observe that the restriction of F to the β_M-fibres of $Mon\,(\Gamma, \mathcal{F})$ is a covering projection over the $\beta-$fibres of Γ. Therefore, the restriction of F to the β_M-fibres of $Mon\,(\Gamma, \mathcal{F})$ is a local diffeomorphism. Then, for each $g \in \Gamma$ the restriction of F induces a isomorphism

$$T_{\epsilon_M(g)} F_{|\beta_M^{-1}(g)} : Ker\left(T_{\epsilon_M(g)}\beta_M\right) \to Ker\left(T_{\epsilon(\beta(g))}\beta\right),$$

where ϵ_M is the section of identities of $Mon\,(\Gamma, \mathcal{F})$. This proves that the β_M-fibres of $Mon\,(\Gamma, \mathcal{F})$ are isomorphic to the $\beta-$fibres of $\overline{\Gamma}$ such that the factorization map

$$\overline{F} : \overline{\Gamma} \to \Gamma,$$

induces an isomorphism between the $\beta-$fibres of $\overline{\Gamma}$ to the $\beta-$fibres of Γ. Hence, it is easy to prove that $\left(Id_M, \overline{F}\right)$ is a morphism of Lie groupoids which induces an isomorphism between the associated Lie algebroids. □

A detailed proof of this result is proved in [72]. The transitive case can be found in [64]. The same methods involved in the construction of the target-simply connected groupoid can be used to prove the following integrability result:

Proposition 2.3.39. *Any Lie subalgebroid of an integrable Lie algebroid is integrable.*

Next, we will deal with the Lie's second fundamental theorem which proves that any morphism of integrable Lie algebroids can be integrated to a unique morphism of the integral Lie groupoids, provided that the domain groupoid is source-simply connected. This result has been proved by K. C. H. Mackenzie and P. Xu [65] (see also [72]).

Theorem 2.3.40 (Lie II). *Let $\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ be Lie groupoids, with $\Gamma_2 \rightrightarrows M_2$ target-simply connected and let $\Phi : A\Gamma_2 \to A\Gamma_1$ be a Lie algebroid morphism over $\phi : M_2 \to M_1$. Then there exists a unique morphism of Lie groupoids $F : \Gamma_2 \to \Gamma_1$ with objects map ϕ which integrates Φ.*

Proof. Let us consider $P = \Gamma_2 \times_{\phi \circ \alpha_2, \alpha_1} \Gamma_1$ be the pullback of $\alpha_1 : \Gamma_1 \to M_1$ along the map $\phi \circ \alpha_2 : \Gamma_2 \to M_1$. We may consider the projection on the first component $pr_1 : P \to \Gamma_2$. Then, for each $(h_2, h_1) \in P$ it is clear that

$$pr_1^{-1}\left(pr_1\left(h_2, h_1\right)\right) = \{h_2\} \times \alpha_1^{-1}\left(\phi\left(\alpha_2\left(h_2\right)\right)\right).$$

Thus, we may construct the foliation on P given by $\mathcal{G} = pr_1^{-1}(\mathcal{F})$, where $\mathcal{F}$ is the foliation of Γ_2 given by the β_2-fibres. It satisfies that $\mathcal{G}$ is a regular foliation such that the dimension of the leaves is $dim\left(\Gamma_1\right) - dim\left(M_1\right) + dim\left(\Gamma_2\right) - dim\left(M_2\right)$ and the tangent spaces at the fibres of $\mathcal{G}$ consist of the vectors $(v_2, v_1) \in T_{h_2}\Gamma_2 \times_{T_{h_2}(\phi \circ \alpha_2), T_{h_1}\alpha_1} T_{h_1}\Gamma_1$ such that $v_2 \in Ker(T_{h_2}\beta_2)$.

(i) $\{T_{h_2}\left(\phi \circ \alpha_2\right)\}\left(v_2\right) = T_{h_1}\alpha_1\left(v_1\right)$.

(ii) $v_2 \in Ker(T_{h_2}\beta_2)$.

Notice that, it is clear that the vertical distribution of pr_1 is

$$(h_2, h_1) \mapsto Ker\left(T_{(h_2,h_1)}pr_1\right) = \{0\} \times Ker\left(T_{h_1}\alpha_1\right).$$

Thus,

$$Ker\left(T_{(h_2,h_1)}pr_1\right) \subseteq T_{(h_2,h_1)}\mathcal{G}\left(h_2, h_1\right),$$

for all $(h_2, h_1) \in P$ with $\mathcal{G}(h_2, h_1)$ the leaf of the foliation $\mathcal{G}$ at (h_2, h_1).
On the other hand, for each $(h_2, h_1) \in P$ we may consider the left translations $L_{h_i} : \beta_i^{-1}(\alpha_i(h_i)) \to \beta_i^{-1}(\beta_i(h_i))$ for $i = 1, 2$. Then, for all $v \in (A\Gamma_2)_{\alpha_2(h_2)} = T_{\epsilon_2(\alpha_2(h_2))}\beta_2^{-1}(\alpha_2(h_2))$, we can take

$$T_{\epsilon_2(\alpha_2(h_2))}L_{h_2}(v) \in Ker\left(T_{h_2}\beta_2\right).$$

Furthermore, using that Φ is a Lie algebroid morphism, for all $v \in (A\Gamma_2)_{\alpha_2(h_2)}$ it makes sense to take

$$T_{\epsilon_1(\alpha_1(h_1))}L_{h_1}(\Phi(v)) \in Ker\left(T_{h_1}\beta_1\right).$$

Next, define the distribution $\mathfrak{H}$ on P, such that the fibres $\mathfrak{H}_{(h_2,h_1)}$ consist of the vectors $\left(T_{\epsilon_2(\alpha_2(h_2))}L_{h_2}(v), T_{\epsilon_1(\alpha_1(h_1))}L_{h_1}(\Phi(v))\right)$ where $v \in (A\Gamma_2)_{\alpha_2(h_2)}$.
Note that, using that Φ is a morphism of Lie algebroids over ϕ ($\Rightarrow \phi \circ \alpha_2 \circ L_{h_2} = \alpha_1 \circ L_{h_1} \circ \Phi$), $\mathfrak{H}$ is well defined. Hence,

$$\mathfrak{H}_{(h_2,h_1)} \subseteq T_{(h_2,h_1)}\mathcal{G}_{(h_2,h_1)}.$$

Since L_{h_2} is a diffeomorphism, the dimension of $\mathfrak{H}_{(h_2,h_1)}$ is equal to $dim(\Gamma_2) - dim(M_2)$. Furthermore, we get that for all $(h_2, h_1) \in P$,

$$Ker\left(T_{(h_2,h_1)}pr_1\right) \cap \mathfrak{H}_{(h_2,h_1)} = \{(0,0)\}.$$

In this way, adding the dimensions, we have

$$T\mathcal{G} = Ker(Tpr_1) \oplus \mathfrak{H}.$$

In fact, we may prove that $\mathfrak{H}$ is a integrable distribution. So, we will denote the folitation integrating $\mathfrak{H}$ by $\mathcal{H}$.
Then, for any $x_2 \in M_2$ the restriction of the projection $(pr_1)_{|\mathcal{H}(\epsilon_2(x_2),\epsilon_1(\phi(x_2)))} : \mathcal{H}(\epsilon_2(x_2), \epsilon_1(\phi(x_2))) \to \alpha^{-1}(x_2)$ is a covering projection. In fact, by taking into account that the β_2-fibres of Γ_2 are simply connected, the map $(pr_1)_{|\mathcal{H}(\epsilon_2(x_2),\epsilon_1(\phi(x_2)))}$ is a diffeomorphism.
Denote by ν_{x_2} the inverse of this diffeomorphism. Now the union of the maps ν_{x_2} gives us a map $\nu : \Gamma_2 \to P$.
Consider the map $F : pr_2 \circ \nu : \Gamma_2 \to \Gamma_1$ where pr_2 is the projection on the

second component $pr_2 : P \to \Gamma_1$. So, F (together with ϕ) gives a morphism of Lie groupoids.
Finally, for any $v \in (A\Gamma_2)_{x_2}$ we have

$$T_{\epsilon_2(x_2)} F(v) = T_{\nu_{x_2}(\epsilon_2(x_2))} pr_2(v, \Phi(v)) = \Phi(v).$$

Hence, the induced map of F over the Lie algebroids is Φ, i.e.,

$$AF = \Phi.$$

□

Using this result we can improve the result in proposition 2.3.39 (see for instance [73]).

Proposition 2.3.41. *Any subalgebroid of an integrable algebroid, $A\Gamma$, is integrable by a unique immersed subgroupoid of the groupoid Γ.*

Let us finish dealing with the third Lie's fundamental theorem.

Lemma 2.3.42. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Then $\Gamma \rightrightarrows M$ is transitive if and only if the associated Lie algebroid $A\Gamma$ is transitive.*

Proof. The key to prove this result is the identity

$$T_{\epsilon(x)}(\mathcal{O}(x)) = \sharp(A\Gamma_x), \tag{2.54}$$

for all $x \in M$, where $A\Gamma_x$ is the fiber through x and $\mathcal{O}(x)$ is the orbit of x (see definition 2.2.10). Observe that, using that α is an open map, $\mathcal{O}(x)$ is a closed subset of M and hence, if $\mathcal{O}(x)$ is an open of M, by connexity, $\mathcal{O}(x) = M$. □

Let A be an integrable transitive Lie algebroid. Then, using the above result, there exists a transitive Lie groupoid $\Gamma \rightrightarrows M$ such that

$$A \cong A\Gamma.$$

Now, taking into account corollary 2.2.28, there exists a principal bundle with structural group G, $\pi : P \to M$, such that

$$A \cong A\left(Gauge\left(P\right)\right) \cong TP/G,$$

where TP/G is the Atiyah algebroid associated with $\pi : P \to M$ (see example 2.3.13). So, we have proved that any integrable transitive Lie algebroid is isomorphic to an Atiyah algebroid.
Notice that the tangent map to π, $T\pi : TP \to TM$, induces an epimorphism

$$\overline{T\pi} : TP/G \to TM,$$

given by,

$$\overline{T\pi}\left(\tau\left(v\right)\right) = T\pi\left(v\right),$$

for all $v \in TP$, where $\tau : TP \to TP/G$ is the quotient projection (observe that the well definition of $\overline{T\pi}$ is given by the fact that π is a principal bundle). Therefore, we have an exact sequence of vector bundles

$$0 \to \left(\mathfrak{g} \times P\right)/G \stackrel{j}{\to} TP/G \stackrel{\overline{T\pi}}{\to} TM \to 0,$$

which is just the *Atiyah sequence associated with the principal bundle* $\pi : P \to M$.

Remark 2.3.43. Suppose that $\mathfrak{g}$ is the Lie algebra of G, $pr_2 : \mathfrak{g} \times P \to P$ is the trivial vector bundle and that the action $\overline{\phi} = (Ad, \phi)$ of G on $\mathfrak{g} \times P$ is given by

$$\left(Ad, \phi\right)\left(g, \left(\xi, p\right)\right) = \left(Ad_g\left(\xi\right), \phi_g\left(p\right)\right),$$

for all $(g, (\xi, p)) \in G \times (\mathfrak{g} \times P)$ where $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$. Note that the space of sections $\Gamma\left(\mathfrak{g} \times P\right)$ of $\mathfrak{g} \times P$ may be identified with the set of $\pi-$vertical vector fields on P. In fact, using that $\pi : P \to M$ is the principal bundle with structural group G,

$$Ker\left(T_p\pi\right) = T_p\left(G \cdot p\right) := \{\xi_P\left(p\right) : \xi \in \mathfrak{g}\},$$

where ξ_P is the infinitesimal generator of ϕ associated to $\xi \in \mathfrak{g}$.
So, it is easy to construct the isomorphism between $\Gamma\left(\mathfrak{g} \times P\right)$ and the set

of $\pi-$vertical vector fields on P. In addition, $\overline{\phi}$ satisfies the conditions (i) and (ii) of example 2.3.12 and the resultant quotient vector bundle $\overline{pr_1} : (\mathfrak{g} \times P)/G \to M = P/G$ is just *the adjoint bundle associated with the principal bundle* $\pi : P \to M$. Furthermore, if for each $\xi \in \mathfrak{g}$, the map

$$\begin{array}{cccc} j: & (\mathfrak{g} \times P)/G & \to & TP/G, \\ & [(\xi, p)] & \mapsto & [\xi_P(p)], \end{array}$$

induces a monomorphism between the vector bundles $(\mathfrak{g} \times P)/G$ and TQ/G.
Thus, $(\mathfrak{g} \times P)/G$ may be considered as a vector subbundle of TQ/G. In addition, the space $\Gamma((\mathfrak{g} \times P)/G)$ may be identified with the set of vector fields on P which are vertical and $G-$invariant. ◇

As a particular case, if $\omega \in \Omega^2(M)$ is a closed $2-$form on M consider the transitive Lie algebroid $A = TM \oplus (M \times \mathbb{R}) \to M$ (see example 2.3.11). Then, the next sequence

$$0 \to M \times \mathbb{R} \overset{i_0}{\to} TM \oplus (M \times \mathbb{R}) \overset{pr_1}{\to} TM \to 0,$$

is a exact sequence, where $i_0 : M \times \mathbb{R} \to TM \oplus (M \times \mathbb{R})$ is defined by

$$i_0(x,t) = 0 \oplus (x,t), \ \forall (x,t) \in M \times \mathbb{R}.$$

Now, denoting by $\Phi : TM \oplus (M \times \mathbb{R}) \to TP/G$ the Lie algebroid isomorphism from A to TP/G then the following diagram

$$\begin{array}{ccc} TM \oplus (M \times \mathbb{R}) & \xrightarrow{pr_1} & TM \\ \Phi \downarrow & & \downarrow Id_{TM} \\ TP/G & \xrightarrow{\overline{T\pi}} & TM \end{array}$$

is conmutative. Hence, using the exact sequences, there exists a smooth map $\varphi : M \times \mathbb{R} \to (\mathfrak{g} \times P)/G$ such that

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{\quad i_0 \quad} & TM \oplus (M \times \mathbb{R}) \\ \varphi \downarrow & & \downarrow \Phi \\ (\mathfrak{g} \times P)/G & \xrightarrow{\quad j \quad} & TP/G \end{array}$$

is a commutative diagram.
On the other hand, consider the map $F : TM \to TP/G$ given by

$$F(v_x) = \Phi(v_x \oplus (x, 0)), \ \forall v_x \in T_x M, \ \forall x \in M.$$

So, identifying $\Gamma(TP/G)$ with the $G-$invariant vector fields, we get a linear map $\cdot^h : \mathfrak{X}(M) \to \mathfrak{X}^G(M)$ and therefore, there exists a connection $\Lambda : TP \to \mathfrak{g}$, such that $\cdot^h$ is the horizontal lifting.
In fact, this connection verifies that

$$Curv^{\Lambda} = \omega. \tag{2.55}$$

Observe that,

$$A(\Gamma_x^x) \cong Ker(\sharp_x) \cong \mathbb{R}.$$

So, given that $G = \Gamma_x^x$, $dim(G) = 1$ and, therefore, we may consider $Curv^{\Lambda}$ as a $2-$form on P as follows

$$Curv^{\Lambda}(v_p, w_p) = pr_2\left(\varphi^{-1}\left(\left[\left(Curv^{\Lambda}(v_p, w_p), p\right)\right]\right)\right),$$

for all $v_p, w_p \in T_p P$ and $p \in P$, where $pr_2 : M \times \mathbb{R} \to \mathbb{R}$ is the projection on the second component. Also, notice that $Curv^{\Lambda}$ can be seen as a $2-$form on M. In fact, using that $Curv^{\Lambda}(v_p, w_p) = 0$, if v_p (or w_p) is vertical. Then, we may define $Curv^{\Lambda} : TM \times_{\pi_M, \pi_M} TM \to \mathfrak{g}$ as follows

$$Curv^{\Lambda}\left(v_{\pi(x)}, w_{\pi(x)}\right) = Curv^{\Lambda}(\overline{v}_x, \overline{w}_x),$$

where $T_x\pi(\overline{v}_x) = v_{\pi(x)}$ and $T_x\pi(\overline{w}_x) = w_{\pi(x)}$. If $T_x\pi(\overline{v}_x) = T_x\pi(\overline{u}_x)$, $\overline{v_x} - \overline{u}_x \in Ker(T_x\pi)$, i.e., $(\overline{v}_x - \overline{u}_x)$ is vertical. Then,

$$Curv^{\Lambda}(\overline{v}_x - \overline{u}_x, \overline{w}_x) = 0,$$

and therefore, $Curv^{\Lambda}$ is well defined. In this way, one can verify identity (2.55).

Now, we are going to use the classical Weil lemma (see [64], theorem 8.1.3).

Theorem 2.3.44. *A closed* $2-$*form* $\omega \in \Omega^2(M)$ *is the curvature of a connection in a* S^1-*bundle if, and only if,*

$$\int_{\gamma} \omega = \int_{S^1} \gamma^* \omega \in \mathbb{Z}, \ \forall \gamma \in \mathcal{C}^{\infty}\left(S^1, M\right). \tag{2.56}$$

In this way, to found a counterexample for the third Lie's fundamental theorem for Lie algebroids we only have to take ω such that (2.56) it is not satisfied. For example, we may take $M = S^2$ together the volume standard form, i.e.,

$$i^* \omega,$$

with $i : S^2 \to \mathbb{R}^3$ the inclusion map and

$$\omega = dx \wedge dy \wedge dz.$$

So, we may found a number k such that

$$\int k i^* \omega,$$

is not a integer number.

Prelude

We have already introduced the necessary tools to present the motivation of our work. Some of the contents included here are collected in [51].
Let us start with an elastic simple material $\mathcal{B}$ with reference configuration ϕ_0. As we have presented, $\mathcal{B}$ has associated a mechanical response $W : \mathcal{B} \times Gl(3, \mathbb{R}) \to V$. Eq. (2.4) shows us that W can be defined on the space of (local) configurations in such a way that for each configuration ϕ we define

$$W\left(j^1_{X,x}\phi\right) = W(X, F),$$

where F is the associated matrix to the $1-$jet at $\phi_0(X)$ of $\phi \circ \phi_0^{-1}$. In fact, composing ϕ_0 by the left, we obtain that W may be equivalently described as a differentiable map $W : \Pi^1(\mathcal{B}, \mathcal{B}) \to V$ from the groupoid of $1-$jets $\Pi^1(\mathcal{B}, \mathcal{B})$ (see example 2.2.9) to the vector space V which does not depend on the image point of the $1-$jets of $\Pi^1(\mathcal{B}, \mathcal{B})$, i.e., for all $X, Y, Z \in \mathcal{B}$

$$W\left(j^1_{X,Y}\phi\right) = W\left(j^1_{X,Z}\left(\phi_0^{-1} \circ \tau_{Z-Y} \circ \phi_0 \circ \phi\right)\right), \tag{2.57}$$

for all $j^1_{X,Y}\phi \in \Pi^1(\mathcal{B}, \mathcal{B})$, where τ_v is the translation map on $\mathbb{R}^3$ by the vector v. Notice that, using Eq. (2.57), we may define W over $\Pi^1\left(\mathcal{B}, \mathbb{R}^3\right)$, which could be seen as an open subset of $\Pi^1\left(\mathbb{R}^3, \mathbb{R}^3\right)$ given by the $1-$jets of local diffeomorphisms from points of $\mathcal{B}$ to points of $\mathbb{R}^3$.
Then, condition of being materially isomorphic is rewritten as follows: *Two material particles X and Y are materially isomorphic if, and only if, there exists a local diffeomorphism ψ from an open neighbourhood $\mathcal{U} \subseteq \mathcal{B}$ of X*

to an open neighbourhood $\mathcal{V} \subseteq \mathcal{B}$ *of* Y *such that* $\psi(X) = Y$ *and*

$$W\left(j^1_{Y,\kappa(Y)}\kappa \cdot j^1_{X,Y}\psi\right) = W\left(j^1_{Y,\kappa(Y)}\kappa\right), \tag{2.58}$$

for all $j^1_{Y,\kappa(Y)}\kappa \in \Pi^1(\mathcal{B},\mathcal{B})$.

For each two points $X, Y \in \mathcal{B}$, we will denote by $G(X,Y)$ the collection of all $1-$jets $j^1_{X,Y}\psi$ which satisfy Eq. (2.2.16), i.e., $G(X,Y)$ is the family of material isomorphisms for X to Y. Remember that, in Section 2.1, we proved that the relation of being "*materially isomorphic*" is an equivalence relation. Indeed, what we proved is that the set $\Omega(\mathcal{B}) = \cup_{X,Y \in \mathcal{B}} G(X,Y)$ may be considered as a groupoid over $\mathcal{B}$ with the composition of $1-$jets as composition law. Thus, $\Omega(\mathcal{B})$ is a subgroupoid of the $1-$jets groupoid $\Pi^1(\mathcal{B},\mathcal{B})$. $\Omega(\mathcal{B})$ is called the *material groupoid* of $\mathcal{B}$.

Notice that, the material symmetry group $G(X)$ at a body point $X \in \mathcal{B}$ is just the isotropy group of $\Omega(\mathcal{B})$ at X. For each $X \in \mathcal{B}$, we will denote the set of material isomorphisms from X to any other point (resp. from any point to X) by $\Omega_X(\mathcal{B})$ (resp. $\Omega^X(\mathcal{B})$). Finally, we will denote the structure maps of $\Omega(\mathcal{B})$ by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\epsilon}$ and $\overline{i}$ which are just the restrictions of the corresponding ones on $\Pi^1(\mathcal{B},\mathcal{B})$.

As a consequence of the continuity of W we have that, for all $X \in \mathcal{B}$, $G(X)$ is a closed subgroup of $\Pi^1(\mathcal{B},\mathcal{B})^X_X$. Hence, the following result is immediate.

Proposition 2.3.45. *Let* $\mathcal{B}$ *be a simple body. Then, for all* $X \in \mathcal{B}$ *the symmetry group* $G(X)$ *is a Lie subgroup of* $\Pi^1(\mathcal{B},\mathcal{B})^X_X$.

This could make us think that $\Omega(\mathcal{B})$ is a Lie subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$. However, this is not true (for instance, the dimensions of the groups of material symmetries could change).

Now, the following result is obvious.

Proposition 2.3.46. *Let* $\mathcal{B}$ *be a body.* $\mathcal{B}$ *is uniform if and only if* $\Omega(\mathcal{B})$ *is a transitive subgroupoid of* $\Pi^1(\mathcal{B},\mathcal{B})$.

Next, by composing appropriately with the reference configuration, smooth uniformity (Definition 2.1.6) may be characterized in the following way.

Proposition 2.3.47. *A body $\mathcal{B}$ is smoothly uniform if, and only if, for each point $X \in \mathcal{B}$ there is an neighbourhood $\mathcal{U}$ around X such that for all $Y \in \mathcal{U}$ and $j^1_{Y,X}\phi \in \Omega(\mathcal{B})$ there exists a local section $\mathcal{P}$ of*

$$\overline{\alpha}_X : \Omega^X(\mathcal{B}) \to \mathcal{B},$$

from $\epsilon(X)$ to $j^1_{Y,X}\phi$.

For obvious reasons, (local) sections of $\overline{\alpha}_X$ will be called *left fields of material isomorphism at X*. On the other hand, local sections of

$$\overline{\beta}^X : \Omega_X(\mathcal{B}) \to \mathcal{B},$$

will be called *right fields of material isomorphism at X*. Thus left (resp. right) fields of material isomorphisms in the sense of Section 2.1 are in a bijective correspondence with these left (resp. right) fields of material isomorphisms via composition with the reference configuration ϕ_0.
Therefore, $\mathcal{B}$ is smoothly uniform if, and only if, for each two points $X, Y \in \mathcal{B}$ there are two open subsets $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$ around X and Y respectively and $\mathcal{P} : \mathcal{U} \times \mathcal{V} \to \Omega(\mathcal{B}) \subseteq \Pi^1(\mathcal{B}, \mathcal{B})$, a differentiable section of the anchor map $\left(\overline{\alpha}, \overline{\beta}\right)$. When $X = Y$ it is easy to realize that we can assume $\mathcal{U} = \mathcal{V}$ and $\mathcal{P}$ is a morphism of groupoids over the identity map, i.e.,

$$\mathcal{P}(Z,T) = \mathcal{P}(R,T)\,\mathcal{P}(Z,R)\,, \ \forall T, R, Z \in \mathcal{U}.$$

So, we may prove a corollary of proposition 2.3.45.

Corollary 2.3.48. *Let $\mathcal{B}$ be a body. $\mathcal{B}$ is smoothly uniform if and only if $\Omega(\mathcal{B})$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.*

Proof. Suppose that $\mathcal{B}$ is smoothly uniform. Fix $j^1_{X,Y}\psi \in \Omega(\mathcal{B})$ and consider $\mathcal{P} : \mathcal{U} \times \mathcal{V} \to \Omega(\mathcal{B})$, a differentiable section of the anchor map $\left(\overline{\alpha}, \overline{\beta}\right)$ with $X \in \mathcal{U}$ and $Y \in \mathcal{V}$. Then, we may construct the following bijection

$$\begin{array}{rccl} \Psi_{\mathcal{U},\mathcal{V}} : & \Omega(\mathcal{U},\mathcal{V}) & \to & \mathcal{B} \times \mathcal{B} \times G(X,Y) \\ & j^1_{Z,T}\phi & \mapsto & \left(Z, T, \mathcal{P}(Z,Y)\left[j^1_{Z,T}\phi\right]^{-1}\mathcal{P}(X,T)\right) \end{array}$$

where $\Omega(\mathcal{U}, \mathcal{V})$ is the set of material isomorphisms from $\mathcal{U}$ to $\mathcal{V}$. By using proposition 2.3.45, we deduce that $G(X, Y)$ is a differentiable manifold. Thus, we can endow $\Omega(\mathcal{B})$ with a differentiable structure of a manifold. Now, the result follows (the converse has been proved in [64]). $\square$

This result clarify even more the difference between smooth uniformity and ordinary uniformity. Furhtermore, it works as an intuition about the lost of differentiability which could have the material groupoid. In particular, as we have previously said, the material groupoid is not necessarily a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$ (see examples in chapter 4). This is a really **important fact** in this memory. Indeed, this is reason because the thesis is divided in **two** parts. First part (chapter 3) is based on the assumption of the material groupoid is a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. Then, we can use its associated Lie algebroid (see Section 2.3) to prove new results associated (above all) the homogeneity of the material [52, 54]. On the other hand, the second part (chapter 4) is focused on attempting of finding new structures to deal with the material groupoid without imposing any condition of differentiability. Thus, it arises the notion of *material distributions* [39, 50, 53] which are generalized to context of general groupoids [51].

Chapter 3

Material algebroid

As we have commented before, this chapter is mainly based on a kind of assumption over the differentiability of the family of material isomorphisms. In particular, this set will have the structure of Lie groupoid. The crucial point about the devolopment of this chapter is the associated Lie algebroid, which is the infinitesimal version of the mentioned Lie groupoid. In fact, since this groupoid encodes the mechanical geometric information of material body, its homogeneity can be characterized through the properties of the Lie algebroid. This is indeed accomplished, and related with the earlier approach developed in [31, 37] in the framework of $G-$structures and second-order non-holonomic $\overline{G}-$structures.

3.1 Simple materials

The content of this section is based in the research article [54] which is part of the new developments presented for fulfillment of the thesis requirement for the degree of Doctor of Philosophy. However, there are also results we have not been published yet. In particular, the content of the subsection entitled *Homogeneity with G-structures*. Here, a serie of results are presented comparing the frame bundle of a (arbitrary) manifold

M with the $1-$jets groupoid of M.

Integrability

As a first step we will introduce the notion of integrability of reduced subgroupoids of the 1-jets groupoid which is going to be closely related with the notion of integrability of $G-$structures (see Appendix A).
Note that there exists a Lie groupoids isomorphism $L : \Pi^1 (\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n \times Gl(n, \mathbb{R})$ over the identity map defined by

$$L\left(j^1_{x,y}\phi\right) = \left(x, y, J\phi_{|x}\right), \ \forall j^1_{x,y}\phi \in \Pi^1 (\mathbb{R}^n, \mathbb{R}^n),$$

where $J\phi_{|x}$ is the Jacobian matrix of ϕ at x. Another way of expressing this isomorphism is identifying $Gl(n, \mathbb{R})$ with the fibre of $F\mathbb{R}^n$ at 0. Then, the isomorphism is given by

$$L\left(j^1_{x,y}\phi\right) = \left(x, y, j^1_{0,0}\left(\tau_{-y} \circ \phi \circ \tau_x\right)\right),$$

for all $j^1_{x,y}\phi \in \Pi^1 (\mathbb{R}^n, \mathbb{R}^n)$, where τ_z denote the translation on $\mathbb{R}^n$ by the vector $z \in \mathbb{R}^n$. So, the inverse map satisfies

$$L^{-1}\left(x, y, j^1_{0,0}\Phi\right) = j^1_{x,y}\left(\tau_y \circ \Phi \circ \tau_{-x}\right),$$

fo all $j^1_{0,0}\Phi \in Gl(n, \mathbb{R})$. Observe that we are canonically identifying any regular matrix with a unique $1-$jet of a local diffeomorphism from 0 to 0. We have thus obtained a Lie groupoid isomorphism $\Pi^1 (\mathbb{R}^n, \mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n \times Gl(n, \mathbb{R})$ over the identity map on $\mathbb{R}^n$. Then, if G is a Lie subgroup of $Gl(n, \mathbb{R})$, we can transport $\mathbb{R}^n \times \mathbb{R}^n \times G$ by this isomorphism to obtain a reduced Lie subgroupoid of $\Pi^1 (\mathbb{R}^n, \mathbb{R}^n)$. These kind of reduced subgroupoids will be called *standard flat* on $\Pi^1 (\mathbb{R}^n, \mathbb{R}^n)$.

Let $U, V \subseteq M$ be two open subsets of M. We denote by $\Pi^1 (U, V)$ the open subset of $\Pi^1 (M, M)$ defined by $(\alpha, \beta)^{-1} (U \times V)$. Note that if $U = V$, then, $\Pi^1 (U, U)$ is in fact the 1-jets groupoid of U and, in this way, our notation is consistent. Furthermore, we are going to think about $\Pi^1 (U, V)$ as the restriction of the Lie groupoid $\Pi^1 (M, M)$ equipped with the restriction of the structure maps (this could not be a Lie groupoid). We will also use this notation for subgroupoids of $\Pi^1 (M, M)$.

Definition 3.1.1. A reduced subgroupoid $\Pi^1_G(M,M)$ of $\Pi^1(M,M)$ will be called *integrable* if it is locally diffeomorphic to the groupoid $\mathbb{R}^n \times \mathbb{R}^n \times G \rightrightarrows \mathbb{R}^n$, for some Lie subgroup G of $Gl(n,\mathbb{R})$.

Before continuing, we need to explain what we understand by "locally diffeomorphic" in this case. So, $\Pi^1_G(M,M)$ is locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times G \rightrightarrows \mathbb{R}^n$ if for all $x, y \in M$ there exist two open sets $U, V \subseteq M$ with $x \in U$, $y \in V$ and two local charts, $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$, which induce a diffeomorphism

$$\Psi_{U,V} : \Pi^1_G(U,V) \to \overline{U} \times \overline{V} \times G, \tag{3.1}$$

such that $\Psi_{U,V} = \left(\psi_U \circ \alpha, \psi_V \circ \beta, \overline{\Psi}_{U,V}\right)$, where

$$\overline{\Psi}_{U,V}\left(j^1_{x,y}\phi\right) = j^1_{0,0}\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right),$$

for all $j^1_{x,y}\phi \in \Pi^1(U,V)$. Notice that, $\Pi^1_G(U,V)$ and $\overline{U} \times \overline{V} \times G$ are Lie groupoids if, and only if, $U = V$ and $\overline{U} = \overline{V}$. Suppose that $U = V$ and $\overline{U} = \overline{V}$, then, for all $x \in U$ $\Psi_{U,U}\left(j^1_{x,x}Id\right) \in G$. However, $\Psi_{U,U}\left(j^1_{x,x}Id\right)$ is not necessarily the identity map and, hence, $\Psi_{U,U}$ is not an isomorphism of Lie groupoids.

Proposition 3.1.2. *Let $\Pi^1_G(M,M)$ be a reduced Lie subgroupoid of $\Pi^1(M,M)$. $\Pi_G(M,M)$ is integrable if, and only if, we can cover M by local charts (ψ_U, U) which induce Lie groupoid isomorphisms from $\Pi^1_G(U,U)$ to the restrictions of the standard flat over G to $\psi_U(U)$.*

Proof. On the one hand, suppose that $\Pi^1(M,M)$ is integrable. Let $x_0 \in M$ be a point in M and $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$ be local charts through x_0 which induced diffeomorphism

$$\Psi_{U,V} : \Pi^1_G(U,V) \to \overline{U} \times \overline{V} \times G.$$

For each $y \in U \cap V$,

$$\overline{\Psi}_{U,V}\left(j^1_{y,y}Id\right) = j^1_{0,0}\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right) \in G.$$

Then, for all $j^1_{x,y}\phi \in \Pi^1_G(U\cap V, U\cap V)$, we have

$$j^1_{0,0}\left(\tau_{-\psi_U(y)}\circ\psi_U\circ\phi\circ\psi_U^{-1}\circ\tau_{\psi_U(x)}\right) =$$
$$= j^1_{0,0}\left(\tau_{-\psi_U(y)}\circ\psi_U\circ\psi_V^{-1}\circ\tau_{\psi_V(y)}\right)\cdot$$
$$\cdot j^1_{0,0}\left(\tau_{-\psi_V(y)}\circ\psi_V\circ\phi\circ\psi_U^{-1}\circ\tau_{\psi_U(x)}\right)\in G.$$

Therefore, denoting $U\cap V$ by W, the map

$$\Psi_{W,W}:\Pi^1_G(W,W)\to\overline{W}\times\overline{W}\times G,$$

is, indeed, a Lie groupoid isomorphism over ψ_W where $\Psi_{W,W} = \left(\psi_W\circ\alpha,\psi_W\circ\beta,\overline{\Psi}_{W,W}\right)$, ψ_W is the restriction of ψ_U to W and for all $j^1_{x,y}\phi\in\Pi^1(W,W)$,

$$\overline{\Psi}_{W,W}\left(j^1_{x,y}\phi\right) = j^1_{0,0}\left(\tau_{-\psi_W(y)}\circ\psi_W\circ\phi\circ\psi_W^{-1}\circ\tau_{\psi_W(y)}\right).$$

On the other hand, suppose that for each $x\in M$ there exists a local chart (ψ_U, U) through x which induces a Lie groupoid isomorphism over ψ_U, namely

$$\Psi_{U,U}:\Pi^1_G(U,U)\to\overline{U}\times\overline{U}\times G, \tag{3.2}$$

such that $\Psi_{U,U} = \left(\psi_U\circ\alpha,\psi_U\circ\beta,\overline{\Psi}_{U,U}\right)$, where for each $j^1_{x,y}\phi\in\Pi^1(U,U)$,

$$\overline{\Psi}_{U,U}\left(j^1_{x,y}\phi\right) = j^1_{0,0}\left(\tau_{-\psi_U(y)}\circ\psi_U\circ\phi\circ\psi_U^{-1}\circ\tau_{\psi_U(x)}\right).$$

Take open sets $U,V\subseteq M$ such that there exist ψ_U and ψ_V satisfy Eq. (3.2). Suppose that $U\cap V\neq\emptyset$. Then, for all $x,y\in U\cap V$, we have

$$j^1_{0,0}\left(\tau_{-\psi_U(y)}\circ\psi_U\circ\psi_V^{-1}\circ\tau_{\psi_V(y)}\right)\cdot \tag{3.3}$$
$$\cdot j^1_{0,0}\left(\tau_{-\psi_V(x)}\circ\psi_V\circ\psi_U^{-1}\circ\tau_{\psi_U(x)}\right)\in G$$

Fixing $z\in U\cap V$, we consider

$$j^1_{0,0}\left(\tau_{-\psi_U(z)}\circ\psi_U\circ\psi_V^{-1}\circ\tau_{\psi_V(z)}\right) = A\in Gl(n,\mathbb{R}).$$

So, we define the diffeomorphism $\overline{\psi}_V = A \cdot \psi_V : V \to A \cdot \overline{V}$. Then, using Eq. (3.3) for all $y \in U \cap V$, we deduce that

$$\begin{aligned} & j^1_{0,0}\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right) = \\ = \quad & j^1_{0,0} A \cdot \left(\tau_{-\psi_V(y)} \circ \psi_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right) \\ = \quad & A \cdot j^1_{0,0}\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right) \in G. \end{aligned} \tag{3.4}$$

In this way, we consider

$$\begin{array}{rccl} \Psi_{U,V}: & \Pi^1_G(U,V) & \to & \overline{U} \times A \cdot \overline{V} \times G \\ & j^1_{x,y}\phi & \mapsto & \left(\psi_U(x), \overline{\psi}_V(y), \overline{\Psi}_{U,V}\left(j^1_{x,y}\phi\right)\right). \end{array}$$

where,

$$\overline{\Psi}_{U,V}\left(j^1_{x,y}\phi\right) = j^1_{0,0}\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right).$$

We will check that $\overline{\Psi}_{U,V}$ is well-defined. We fix $j^1_{x,y}\phi \in \Pi^1_G(U,V)$. Then, we can consider two cases:

(i) $y \in U \cap V$. Then, using Eq. (3.4)

$$\begin{aligned} & j^1_{0,0}\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right) = \\ & = j^1_{0,0}\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right) \cdot \\ & \cdot j^1_{0,0}\left(\tau_{-\psi_U(y)} \circ \psi_U \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right) \in G. \end{aligned}$$

(ii) $y \notin U \cap V$. Then,

$$j^1_{z,x}\left(\psi_V^{-1} \circ \tau_{\psi_V(z) - \psi_V(y)} \circ \psi_V \circ \phi\right) = j^1_{z,x}\phi_z,$$

which is in $\Pi^1_G(M,M)$. Hence,

$$j^1_{0,0}\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right) =$$

$$= A \cdot j^1_{0,0}\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right)$$

$$= A \cdot j^1_{0,0}\left(\tau_{-\psi_V(z)} \circ \psi_V \circ \phi_z \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right)$$

$$= j^1_{0,0}\left(\tau_{-\overline{\psi}_V(z)} \circ \overline{\psi}_V \circ \phi_z \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right) \in G.$$

Thus, it is immediate to prove that $\Psi_{U,V}$ is a diffeomorphism which commutes with the restrictions of the structure maps.

Finally, if $U \cap V = \emptyset$ we can find a finite family of local neighbourhoods $\{V_i\}_{i=1,\ldots,k}$ such that

(i) $U = V_1$

(ii) $V = V_k$

(iii) $V_i \cap V_{i+1} \neq \emptyset, \ \forall i$

Thus, we can find $\Psi_{U,V}$ following a similar procedure to the one used above.

□

Remark 3.1.3. Let $\Pi^1_G(M,M)$ be an integrable subgroupoid of $\Pi^1(M,M)$ by the Lie subgroup G of $Gl(n,\mathbb{R})$, i.e., locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times G$. Suppose that there exists another subgroup of $Gl(n,\mathbb{R})$, $\tilde{G}$, such that $\Pi^1_G(M,M)$ is locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times \tilde{G}$. Then, using the above result, it is easy to see that G and $\tilde{G}$ are conjugated subgroups of $Gl(n,\mathbb{R})$. Conversely, if G and $\tilde{G}$ are conjugated subgroups of $Gl(n,\mathbb{R})$ then, $\Pi^1_G(M,M)$ is locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times G$ if and only if $\Pi^1_G(M,M)$ locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times \tilde{G}$.

There is a special reduced subgroupoid of $\Pi^1(M,M)$ which will play an important role in the following. A trivial reduced subgroupoid of $\Pi^1(M,M)$ or *parallelism of* $\Pi^1(M,M)$ is a reduced subgroupoid of $\Pi^1(M,M)$, $\Pi^1_e(M,M) \rightrightarrows M$, such that for each $x,y \in M$ there exists a unique $1-$jet $j^1_{x,y}\phi \in \Pi^1_e(M,M)$.

So, having a trivial reduced subgroupoid of $\Pi^1(M,M)$, $\Pi^1_e(M,M)$, we can consider a map $\mathcal{P} : M \times M \to \Pi^1(M,M)$ such that $\mathcal{P}(x,y)$ is the unique $1-$jet from x to y which is in $\Pi^1_e(M,M)$. It is easy to prove that $\mathcal{P}$ is, indeed, a global section of (α,β). Conversely, every global section of (α,β) (understanding "section" as section in the category of Lie groupoids, i.e., Lie groupoid morphism from the pair groupoid $M \times M$ to $\Pi^1(M,M)$ which is a section of the morphism (α,β)) can be seen as a parallelism of $\Pi^1(M,M)$. Using this, we can also speak about *integrable sections of* (α,β).
Now, using the induced coordinates given in Eq. (2.25)

$$\Pi^1(U,V) : \left(x^i, y^j, y^j_i\right), \tag{3.5}$$

an integrable section can be written locally as follows,

$$\mathcal{P}\left(x^i, y^j\right) = \left(x^i, y^j, \delta^j_i\right),$$

or equivalently

$$\mathcal{P}(x,y) = j^1_{x,y}\left(\psi^{-1} \circ \tau_{\psi(y)-\varphi(x)} \circ \varphi\right), \tag{3.6}$$

for some two local charts $(\varphi,U), (\psi,V)$ on M.
Notice that, using proposition 3.1.2, $\mathcal{P}$ is an integrable section if, and only if, we can cover M by local charts (φ,U) such that

$$\mathcal{P}_{|U}(x,y) = j^1_{x,y}\left(\varphi^{-1} \circ \tau_{\varphi(y)-\varphi(x)} \circ \varphi\right). \tag{3.7}$$

Next, analogously to the case of $G-$structures, we can characterize the integrable subgroupoids using (local) integrable sections (see proposition A.0.9). However, in this case it is not so easy because, having a reduced subgroupoid, we do not know anything about the structure group G. So, firstly, we will have to solve this problem. Let $\Pi^1_G(M,M)$ be a reduced subgroupoid of $\Pi^1(M,M)$ and $\overline{z}_0 \in FM$ be a frame at $z_0 \in M$. Then, we define

$$G := \{\overline{z}_0^{-1} \cdot g \cdot \overline{z}_0 \;/\; g \in \Pi^{1\ z_0}_{G\ z_0}\} = \overline{z}_0^{-1} \cdot \Pi^{1\ z_0}_{G\ z_0} \cdot \overline{z}_0, \tag{3.8}$$

where $\Pi^{1\ z_0}_{G\ z_0}$ is the isotropy group of $\Pi^1(M,M)$ at z_0. Therefore, G is a Lie subgroup of $Gl(n,\mathbb{R})$. This Lie group will be called *associated Lie group* to $\Pi^1_G(M, Mtfy)$.

Note that, as a difference with $G-$structures, we do not have a unique Lie group G. In fact, let $\overline{y}_0$ be a frame at y_0 and $\tilde{G}$ be the associated Lie group, then, if we take $L_{z_0,y_0} \in \Pi^1_G(z_0, y_0)$ we have

$$G = [\overline{y}_0^{-1} \cdot L_{z_0,y_0} \cdot \overline{z}_0]^{-1} \cdot \tilde{G} \cdot [\overline{y}_0^{-1} \cdot L_{z_0,y_0} \cdot \overline{z}_0],$$

i.e., G and $\tilde{G}$ are conjugated subgroups of $Gl(n, \mathbb{R})$. Notice that this fact is what we have expected because of remark 3.1.3.

Proposition 3.1.4. *A reduced subgroupoid $\Pi^1_G(M, M)$ of $\Pi^1(M, M)$ is integrable if and only if for each two points $x, y \in M$ there exist coordinate systems (x^i) and (y^j) over $U, V \subseteq M$, respectively with $x \in U$ and $y \in V$ such that the local section,*

$$\mathcal{P}(x^i, y^j) = \left(x^i, y^j, \delta^j_i\right), \tag{3.9}$$

takes values into $\Pi^1_G(M, M)$.

Proof. First, it is obvious that if $\Pi^1_G(M, M)$ is integrable then, we can restrict the maps $\Psi^{-1}_{U,V}$ to $\overline{U} \times \overline{V} \times \{e\}$ to get (local) integrable sections of (α, β) which takes values on $\Pi^1_G(M, M)$.
Conversely, in a similar way to proposition 3.1.2 we can claim that for each $x \in M$ there exists an open set $U \subseteq M$ with $x \in U$ and $\mathcal{P} : U \times U \to \Pi^1_G(U, U)$ an integrable sections of (α, β) given by

$$\mathcal{P}(x, y) = j^1_{x,y}\left(\psi^{-1}_U \circ \tau_{\psi_U(y) - \psi_U(x)} \circ \psi_U\right),$$

where $\psi_U : U \to \overline{U}$ is a local chart at x.
Then, we can build the map

$$\Psi^{-1}_{U,U} : \overline{U} \times \overline{U} \times \{e\} \to \Pi^1_G(U, U),$$

defined in the obvious way.
Now, let $z_0 \in U$ be a point at U, $\overline{z}_0 = j^1_{0,z_0}\left(\psi^{-1}_U \circ \tau_{\psi_U(z_0)}\right) \in FU$ be a

frame at z_0 and G be the Lie subgroup satisfying Eq. (3.8). Then, we can define

$$\Psi_{U,U} : \Pi^1_G(U,U) \to \overline{U} \times \overline{U} \times G,$$

where for each $j^1_{z_0,z_0}\phi \in \Pi^1_G(z_0)$ and $x, y \in \overline{U}$ we define

$$\Psi^{-1}_{U,U}\left(x, y, Z_0^{-1} \cdot j^1_{z_0,z_0}\phi \cdot Z_0\right) =$$

$$= j^1_{0,\psi_U^{-1}(y)}\left(\psi_U^{-1} \circ \tau_y\right) \cdot [\overline{z}_0^{-1} \cdot j^1_{z_0,z_0}\phi \cdot \overline{z}_0] \cdot j^1_{\psi_U^{-1}(x),0}(\tau_{-x} \circ \psi_U).$$

Hence the map $\Psi_{U,U} : \overline{U} \times \overline{U} \times G \to \Pi^1_G(U,U)$ is an isomorphism of Lie groupoids induced by ψ_U.
To end the proof, we only have to use proposition 3.1.2. □

Let $\mathcal{B}$ be a body. Taking into account the definition of homogeneity (see definition 2.1.10) and the above result we can give the following proposition:

Proposition 3.1.5. *Let $\mathcal{B}$ be a uniform body. If $\mathcal{B}$ is homogeneous then $\Omega(\mathcal{B})$ is integrable. Conversely, $\Omega(\mathcal{B})$ is integrable implies that $\mathcal{B}$ is locally homogeneous.*

Now, we want to work with the notion of integrability in the associated Lie algebroid of the 1-jets groupoid. So, we will introduce this notion and relate it with the integrability of reduced subgroupoids of $\Pi^1(M,M)$.
Note that the induced map of the Lie groupoid isomorphism $L : \Pi^1(\mathbb{R}^n,\mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n \times Gl(n,\mathbb{R})$ is given by a Lie algebroid isomorphism

$$AL : A\Pi^1(\mathbb{R}^n,\mathbb{R}^n) \to T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{gl}(n,\mathbb{R})),$$

where $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{gl}(n,\mathbb{R}))$ is the trivial Lie algebroid on $\mathbb{R}^n$ with structure algebra $\mathfrak{gl}(n,\mathbb{R})$ (see example 2.3.8).
Now, if $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$, we can transport $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{g})$ by this isomorphism to obtain a reduced Lie subalgebroid of $A\Pi^1(\mathbb{R}^n,\mathbb{R}^n)$. These kind of reduced subalgebroids will be called *standard flat* on $A\Pi^1(\mathbb{R}^n,\mathbb{R}^n)$.

Let $U \subseteq M$ be an open subset of M. We denote by $A\Pi^1(U,U)$ the open Lie subalgebroid of $A\Pi^1(M,M)$ defined by the associated Lie algebroid of $\Pi^1(U,U)$.

Definition 3.1.6. Let $A\Pi^1_G(M,M)$ be a Lie subalgebroid of $A\Pi^1(M,M)$. $A\Pi^1_G(M,M)$ is said to be *integrable by* G if it is locally isomorphic to the algebroid $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the Lie subgroup G of $Gl(n,\mathbb{R})$.

Again, we need to explain what we understand by "locally isomorphic" in this case. $A\Pi^1_G(M,M)$ is locally diffeomorphic to $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{g})$ if for all $x \in M$ there exists an open set $U \subseteq M$ with $x \in U$ and a local chart, $\psi_U : U \to \overline{U}$, which induces an isomorphism of Lie algebroids,

$$A\Psi_{U,U} : A\Pi^1_G(U,U) \to T\mathbb{R}^n \oplus (\mathbb{R}^n \times \mathfrak{g}), \tag{3.10}$$

such that $A\Psi_{U,U}$ is the induced map of the following isomorphism of Lie groupoids

$$\Psi_{U,U} : \Pi^1_G(U,U) \to \overline{U} \times \overline{U} \times G,$$

for some Lie subgroupoid $\Pi^1_G(U,U)$ of $\Pi^1(U,U)$, where for all $j^1_{x,y}\phi \in \Pi^1_G(U,U)$ the image $\Psi_{U,U}\left(j^1_{x,y}\phi\right)$ is given by

$$\left(\psi_U(x), \psi_U(y), j^1_{0,0}\left(\tau_{-\psi_U(y)} \circ \psi_U \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right), \tag{3.11}$$

So, for each open $U \subseteq M$, $A\Pi^1_G(U,U)$ is integrable by a Lie subgroupoid $\Pi^1_G(U,U)$ of $\Pi^1(U,U)$. Using the uniqueness of integrating immersed (source-connected) subgroupoids (see proposition 2.3.41), $A\Pi^1_G(M,M)$ is integrable by a Lie subgroupoid of $\Pi^1(M,M)$ which will be denoted by $\Pi^1_G(M,M)$.

By definition, it is immediate to prove that $A\Pi_G(M,M)$ is integrable by G if and only if, $\Pi_G(M,M)$ is integrable (by using proposition 3.1.2).

Analogously to the case of $1-$jets groupoid, a *parallelism* of $A\Pi^1(M,M)$ is an associated Lie algebroid of a parallelism of $\Pi^1(M,M)$. Hence, using the Lie's second fundamental theorem 2.3.40, a parallelism is a section of $\sharp$ (understanding "section" as section in the category of Lie algebroids, i.e., Lie algebroid morphism from the tangent algebroid TM to $A\Pi^1(M,M)$ which is a section of the morphism $\sharp$) and reciprocally. In this way, we will also speak about *integrable sections of* $\sharp$.

Let $\left(x^i\right)$ be a local coordinate system defined on some open subset $U \subseteq M$. Then, we will use the local coordinate system defined in Eq. (2.48),

$$A\Pi^1\left(U,U\right):\left(\left(x^i,x^i,\delta^i_j\right),0,v^i,v^i_j\right)\cong\left(x^i,v^i,v^i_j\right),$$

which are, indeed, induced coordinates by the functor A from local coordinates on $\Pi^1\left(U,U\right)$.
Notice that each integrable section of (α,β) in $\Pi^1\left(M,M\right)$, $\mathcal{P}$, is a Lie groupoid morphism. Hence, $\mathcal{P}$ induces a Lie algebroid morphism $A\mathcal{P}: TM \to A\Pi^1\left(M,M\right)$ (see theorem 2.3.26) which is a section of $\sharp$ and is given by

$$A\mathcal{P}\left(v_x\right)=T_x\mathcal{P}_x\left(v_x\right),\ \forall v_x\in T_xM,$$

where $\mathcal{P}_x : M \to \Pi^1_x\left(M,M\right)$ satisfies that

$$\mathcal{P}_x\left(y\right)=\mathcal{P}\left(x,y\right),\forall x,y\in M.$$

So, taking into account that, locally,

$$\mathcal{P}\left(x^i,y^j\right)=\left(x^i,y^j,\delta^j_i\right),$$

we have that each integrable section can be written locally as follows

$$A\mathcal{P}\left(x^i,\frac{\partial}{\partial x^i}\right)=\left(x^i,\frac{\partial}{\partial x^i},0\right).$$

Now, using proposition 3.1.4, we have the following analogous proposition.

Proposition 3.1.7. *A reduced subalgebroid* $A\Pi^1_G\left(M,M\right)$ *of* $A\Pi^1\left(M,M\right)$ *is integrable by* G *if and only if there exist local integrable sections of* $\sharp$ *covering* M *which takes values on* $A\Pi^1_G\left(M,M\right)$*.*

Equivalently, for each point $x\in M$ there exists a local coordinate system $\left(x^i\right)$ over an open set $U\subseteq M$ with $x\in U$ such that the local sections

$$\Delta\left(x^i,\frac{\partial}{\partial x^i}\right)=\left(x^i,\frac{\partial}{\partial x^i},0\right),$$

are in $A\Pi^1_G(M,M)$.

Finally, we will use the algebroid of derivations on M. Thus, as we have shown in section 2.3, the map $\mathcal{D} : \Gamma\left(A\Pi^1(M,M)\right) \to Der(TM)$ given by

$$\mathcal{D}(\Lambda) = D^{\Lambda} = D^{\Lambda} = \frac{\partial}{\partial t_{|0}}\left(\varphi_t^{\Lambda}\right)^*,$$

defines a Lie algebroid isomorphism $\mathcal{D} : A\Pi^1(M,M) \to \mathfrak{D}(TM)$ over the identity map on M.

Let Δ be a linear section of $\sharp$ in $A\Pi^1(M,M)$. Then, $\mathcal{D}$ induces a covariant derivative on M, ∇^{Δ}. Thus, for each $\left(x^i\right)$ local coordinate system on M,

$$\Delta\left(x^i, \frac{\partial}{\partial x^j}\right) = \left(x^i, \frac{\partial}{\partial x^j}, \Delta^k_{i,j}\right).$$

Hence, remember that the functions $\Delta^k_{i,j}$ are the Christoffel symbols of ∇^{Δ}, i.e.,

$$\nabla^{\Delta}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Delta^k_{i,j}\frac{\partial}{\partial x^k}.$$

With this fact in mind, we can give another characterization of the integrability over the $1-$jets algebroid.

Proposition 3.1.8. *Let Δ be a linear section of $\sharp$ in the $1-$jets Lie algebroid, $A\Pi^1(M,M)$. Then, it is integrable if, and only if, for each point $x \in M$ there exists a local coordinate system $\left(x^i\right)$ on an open set $U \subseteq M$ with $x \in U$ such that ∇^{Δ} is a covariant derivative with Christoffel symbols equal to zero, i.e.,*

$$\nabla^{\Delta}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0, \ \forall i,j.$$

In other words, integrable linear section of $\sharp$ coincide with the torsion-free and flat connections (see Box 2.1, lemma 2.1.9).

Let $W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ be the mechanical response which defines $\Omega(\mathcal{B})$. Consider a section $\Lambda \in \Gamma(A\Omega(\mathcal{B}))$. So, the flow of the left-invariant vector field Θ^Λ, $\{\varphi_t^\Lambda\}$, can be restricted to $\Omega(\mathcal{B})$.
Hence, we have

$$\begin{aligned} W\left(\varphi_t^\Lambda(g)\right) &= W\left(\varphi_t^\Lambda(g \cdot \overline{\epsilon}(\overline{\alpha}(g)))\right) \\ &= W\left(g \cdot \varphi_t^\Lambda(\overline{\epsilon}(\overline{\alpha}(g)))\right) \\ &= W(g) \end{aligned}$$

for all $g \in \Pi^1(\mathcal{B},\mathcal{B})$. Thus, for each $g \in \Pi^1(\mathcal{B},\mathcal{B})$, we deduce

$$TW\left(\Theta^\Lambda(g)\right) = \frac{\partial}{\partial t_{|0}}\left(W\left(\varphi_t^\Lambda(g)\right)\right) = \frac{\partial}{\partial t_{|0}}(W(g)) = 0.$$

Therefore,

$$TW\left(\Theta^\Lambda\right) = 0. \tag{3.12}$$

Conversely, it is easy to prove that **(1)'** and **(2)'** imply that $\Theta \in \Gamma\left(A\overline{\Omega}(\mathcal{B})\right)$.
In this way, the material algebroid can be defined without using material groupoid by imposing Eq. (3.12). Thus, we can characterize the homogeneity and uniformity using the material Lie algebroid.
Now, using the above results we can give the following results.

Proposition 3.1.9. *Let $\mathcal{B}$ be a uniform body. If $\mathcal{B}$ is homogeneous, then, $A\Omega(\mathcal{B})$ is integrable by a Lie subgroup G of $Gl(n,\mathbb{R})$. Conversely, if $A\Omega(\mathcal{B})$ is integrable by G then $\mathcal{B}$ is locally homogeneous.*

Using proposition 3.1.7, this result can be expressed locally as follows.

Proposition 3.1.10. *Let $\mathcal{B}$ be a uniform body. $\mathcal{B}$ is locally homogeneous if and only if for each point $x \in \mathcal{B}$ there exists a local coordinate system $\left(x^i\right)$ over $U \subseteq \mathcal{B}$ with $x \in U$ such that the local section of $\sharp$,*

$$\Delta\left(x^i, \frac{\partial}{\partial x^i}\right) = \left(x^i, \frac{\partial}{\partial x^i}, 0\right),$$

takes values in $A\Omega(\mathcal{B})$.

Finally, denoting by $\mathcal{D}(\mathcal{B})$ to the Lie subalgebroid of the derivation algebroid on $\mathcal{B}$, $\mathcal{D}(A\Omega(\mathcal{B})) \leq \mathfrak{D}(T\mathcal{B})$, we can give the following result:

Theorem 3.1.11. *Let $\mathcal{B}$ be a uniform body. If $\mathcal{B}$ is homogeneous respect to the global deformation κ, there exists a global covariant derivative on $\mathcal{B}$ which takes values in $\mathcal{D}(\mathcal{B})$ and is trivial respect to κ.*
Conversely, $\mathcal{B}$ is locally homogeneous if and only if for each point $x \in \mathcal{B}$ there exists a local coordinate systems $\left(x^i\right)$ over $U \subseteq \mathcal{B}$ with $x \in U$ such that the local covariant derivative on $\mathcal{B}$ characterized by,

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0, \ \forall i, j,$$

which takes values in $\mathcal{D}(\mathcal{B})$.

Roughly speaking, $\mathcal{B}$ is locally homogeneous if, and only if, there exist local trivial covariant derivatives on $\mathcal{B}$ which take values in $\mathcal{D}(\mathcal{B})$.

Remark 3.1.12. There is still another interesting way of interpreting the $1-$jets Lie groupoid on a body $\mathcal{B}$ (and, hence, of interpreting the integrability of a reduced subgroupoid $\Pi^1_G(\mathcal{B}, \mathcal{B})$ of $\Pi^1(\mathcal{B}, \mathcal{B})$). As we know, there exists another structure of Lie groupoid related with $F\mathcal{B}$, the Gauge groupoid of the principal bundle $F\mathcal{B}$ (see example 2.2.21).

We only have to take into account that

$$Gauge\,(F\mathcal{B}) \cong \mathcal{B} \times F\mathcal{B}.$$

Furthermore, translating points we can construct an isomorphism of Lie groupoid from $\mathcal{B} \times F\mathcal{B}$ to $\Pi^1(\mathcal{B}, \mathcal{B})$ (notice that this isomorphism depends on the reference configuration ϕ_0). Thus, the $1-$jets Lie groupoid can be seen as the gauge groupoid of the principal bundle $F\mathcal{B}$ and, therefore, the $1-$jets Lie algebroid can be seen as the Atiyah algebroid associated with $F\mathcal{B}$.

Remark 3.1.13. Notice that, using Eq. (3.12) we can characterize the Lie subalgebroid $\mathcal{D}(\mathcal{B})$ of the derivation algebroid on $\mathcal{B}$ by the derivations on $\mathcal{B}$, D, such that the associated section of $A\Pi^1(\mathcal{B}, \mathcal{B})$ satisfies Eq. (3.12).

Let $\left(x^i\right)$ be a local coordinate system on $\mathcal{B}$ and D be a derivation on $\mathcal{B}$ with base vector field Θ. We denote

- $\Theta\left(x^{i}\right)=\left(x^{i},\Theta^{j}\right)$.
- $D\left(\frac{\partial}{\partial x^{i}}\right)=\Theta_{i}^{j}\frac{\partial}{\partial x^{j}}$.

Then, D is in $\mathcal{D}\left(\mathcal{B}\right)$ if and only if over any $\left(x^{i}\right)$ local coordinate system on $\mathcal{B}$ it is satisfied that

$$dW_{|\left(x^{i},x^{i},g_{i}^{j}\right)}\left(0,\Theta^{j},g_{l}^{j}\cdot\Theta_{i}^{l}\right)=0,$$

for all material symmetry $g\in G\left(x\right)$ which is locally written as follows

$$g\cong\left(x^{i},x^{i},g_{i}^{j}\right).$$

Homogeneity with G-structures

Finally, we will prove that our definition of homogeneity (see definition 2.1.10 or proposition 3.1.5) is, indeed, equivalent to that used in [31] where the authors use $G-$structures to characterize this property (see Definition 2.1.13).

Let $\overline{Z}_{0}$ be a fixed frame at $Z_{0}\in\mathcal{B}$. Then, we construct a $G_{0}-$structure $\omega_{G_{0}}\left(\mathcal{B}\right)$ on $\mathcal{B}$ containing $\overline{Z}_{0}$ given by,

$$\omega_{G_{0}}\left(\mathcal{B}\right)=\Omega_{Z_{0}}\left(\mathcal{B}\right)\cdot\overline{Z}_{0}.$$

So, proposition 2.1.14 shows us that the (local) homogeneity with respect to $\overline{Z}_{0}$ is equivalent to the integrability of $\omega_{G_{0}}\left(\mathcal{B}\right)$.
To compare both defintions, we will start constructing the following map:

$$\begin{array}{rccc}\mathcal{G}: & \Gamma\left(FM\right) & \to & \Gamma_{\left(\alpha,\beta\right)}\left(\Pi^{1}\left(M,M\right)\right)\\ & P & \mapsto & \mathcal{G}P.\end{array}$$

such that,

$$\mathcal{G}P\left(x,y\right)=P\left(y\right)\cdot\left[P\left(x\right)^{-1}\right]\tag{3.13}$$

where $\cdot$ is the composition of $1-$jets. Obviously, $\mathcal{G}$ is well-defined.
Before starting to work with integrable sections we are interested in

dilucidating when an element of $\Gamma_{(\alpha,\beta)}\left(\Pi^1(M,M)\right)$ can be inverted by $\mathcal{G}$. First, we consider $P \in \Gamma(FM)$; then for all $x, y, z \in M$, we have

$$\mathcal{G}P(y,z) \cdot \mathcal{G}P(x,y) = \mathcal{G}P(x,z), \tag{3.14}$$

i.e., $\mathcal{G}P$ is a morphism of Lie groupoids over the identity map on M from the pair groupoid $M \times M$ to $\Pi^1(M,M)$. Therefore, not every element of $\Gamma_{(\alpha,\beta)}\left(\Pi^1(M,M)\right)$ can be inverted by $\mathcal{G}$ but we can prove the following result.

Proposition 3.1.14. *Let $\mathcal{P}$ be a section of (α,β) in $\Pi^1(M,M)$. Then there exists P a section of FM such that*

$$\mathcal{G}P = \mathcal{P},$$

if, and only if, $\mathcal{P}$ is a morphism of Lie groupoids over the identity map from the pair groupoid $M \times M$ to $\Pi^1(M,M)$.

Proof. We have proved the right implication. Conversely, if Eq. (3.14) is satisfied we can define $P \in \Gamma(FM)$ as follows

$$P(x) = \mathcal{P}(z,x) \cdot j^1_{0,z}\psi,$$

where $j^1_{0,z}\psi \in FM$ is fixed. Then, using Eq. (3.14), we have

$$\mathcal{G}P = \mathcal{P}.$$

□

However, there is not a unique P such that $\mathcal{G}P = \mathcal{P}$. We will study this problem in remark 3.1.15. Notice that the relevant sections of (α,β) are going to be the parallelisms which are, indeed, the morphisms of Lie groupoids over the identity map from the pair groupoid $M \times M$ to $\Pi^1(M,M)$.

Next, suppose that P is an integrable section of FM. Then, for each point $x \in M$ there exists a local coordinate system $\left(x^i\right)$ on M such that

$$P\left(x^i\right) = \left(x^i, \delta^i_j\right),$$

or equivalently,

$$P(x) = j^1_{0,x}\left(\varphi^{-1} \circ \tau_{\varphi(x)}\right), \tag{3.15}$$

where φ is the local chart over x and $\tau_{\varphi(x)}$ denote the translation on $\mathbb{R}^n$ by the vector $\varphi(x)$.

Then, for all $x, y \in M$ there exist two charts ψ and φ over x and y respectively such that

$$\mathcal{G}P(x, y) = j^1_{x,y}\left(\psi^{-1} \circ \tau_{\psi(y)-\varphi(x)} \circ \varphi\right), \tag{3.16}$$

i.e., $\mathcal{G}P$ is an integrable section of (α, β) on $\Pi^1(M, M)$. Hence $\mathcal{G}$ takes integrable sections on FM into integrable sections in $\Pi^1(M, M)$. Furthermore, using Eq. (3.6), for each integrable section of (α, β) $\mathcal{P}$ in $\Pi^1(M, M)$ we can construct P, integrable section on FM such that

$$\mathcal{G}P = \mathcal{P}.$$

However, this fact does not implies that P integrable is equivalent to $\mathcal{G}P$ integrable. So, we will study this problem in the following remark.

Remark 3.1.15. Let $P, Q : M \to FM$ be two sections of FM such that $\mathcal{G}P = \mathcal{G}Q$, i.e., for all $x, y \in M$, we have

$$P(y) \cdot [P(x)^{-1}] = Q(y) \cdot [Q(x)]^{-1}.$$

Then,

$$[Q(y)]^{-1} \cdot P(y) = [Q(x)]^{-1} \cdot P(x).$$

So, denoting by $Z_0 = [Q(x)]^{-1} \cdot P(x)$, we deduce that

$$P(x) = Q(x) \cdot Z_0.$$

Conversely, for each $1-$jet $Z_0 = j^1_{0,0}\phi \in F\mathbb{R}^n_0$, where $F\mathbb{R}^n_0$ is the fibre of $F\mathbb{R}^n$ over 0, and each section of FM, $P : M \to FM$, the section of FM given by

$$Q(x) = P(x) \cdot Z_0, \tag{3.17}$$

satisfies that

$$\mathcal{G}P = \mathcal{G}Q.$$

Thus, we have shown that for each section on FM $P : M \to FM$

$$\mathcal{G}^{-1}(\mathcal{G}P) = \{P \cdot Z_0 \ / \ Z_0 \in F\mathbb{R}^n_0\},$$

i.e., the map $\mathcal{G}$ can be considered as an injective map over the quotient space by Eq. (3.17).

Using this, it is obvious that, if $\mathcal{G}P$ is integrable, then, P is integrable too, i.e., the map $\mathcal{G}$ restricted to the integrable sections can be considered as a one-to-one map over the quotient space by Eq. (3.17).

Finally, we can generalize the map $\mathcal{G}$ into a map which takes $G-$structures on M into reduced subgroupoids of $\Pi^1(M,M)$. Let $\omega_G(M)$ be a $G-$structure on M, then we consider the following set,

$$\mathcal{G}(\omega_G(M)) = \{L_y \cdot [L_x^{-1}] \ / \ L_x, L_y \in \omega_G(M)\}.$$

It is straightforward to prove that $\mathcal{G}(\omega_G(M))$ is a reduced subgroupoid of $\Pi^1(M,M)$. In fact, taking a local section of $\omega_G(M)$,

$$P_U : U \to \omega_G(U),$$

the map given by

$$\begin{array}{rccl} F_U : & \Pi^1(U,U) & \to & FU \\ & L_{x,y} & \mapsto & L_{x,y} \cdot [P_U(x)] \end{array}$$

is a diffeomorphism which satisfies that $F_U\left(\Pi^1_G(U,U)\right) = \omega_G(U)$.

Analogously to parallelisms, we can prove that every reduced subgroupoid can be inverted by $\mathcal{G}$ into a $G-$structure on M, where G is defined by Eq. (3.8) with $Z_0 \in FM$ fixed.

We consider $z_0 = \pi_M(Z_0)$. Then, we can generate a $G-$structure over M in the following way

$$\omega_G(M) := \{L_{z_0,x} \cdot Z_0 \cdot g \ / \ g \in G, \ L_{z_0,x} \in \Pi^1_G(M,M)_{z_0}\}.$$

Notice that the fibre of $\omega_G(M)$ at $x \in M$ is given by the set

$$\{L_{z_0,x} \cdot Z_0 \cdot g \;/\; g \in G\},$$

for any fixed $L_{z_0,x} \in \Pi^1_G(M,M)_{z_0}$. In fact, for two $L_{z_0,x}, G_{z_0,x} \in \Pi^1_G(M,M)_{z_0}$,

$$[L_{z_0,x} \cdot Z_0]^{-1} \cdot G_{z_0,x} \cdot Z_0 \in G.$$

Notice that the map $L_{z_0,x} \to L_{z_0,x} \cdot Z_0$ defines an isomorphism of principal bundles from $\Pi^1_G(M,M)_{z_0}$ to $\omega_G(M)$.

Finally, let $\omega_G(M)$ be an integrable $G-$structure on M, using proposition A.0.9 and Eq. (3.15) for each point $x \in M$ there exists a local chart (φ, U) with $x \in U$ such that $\omega_G(U)$ is given by the $1-$jets $j^1_{0,x}\left(\varphi^{-1} \circ \tau_{\varphi(x)}\right) \cdot A$ for all $x \in U$ and $A \in G$.
Therefore, taking two local charts (φ, U) and (ψ, V) and denoting $\mathcal{G}(\omega_G(M))$ by $\Pi^1_G(M,M)$ we have that the elements of $\Pi^1_G(U,V)$ are given by

$$j^1_{0,y}\left(\psi^{-1} \circ \tau_{\psi(y)}\right) \cdot A \cdot j^1_{x,0}\left(\tau_{-\varphi(x)} \circ \varphi\right) \tag{3.18}$$

for all $x \in U$, $y \in V$, $A \in G$. So, the local section of (α, β) given by $j^1_{x,y}\left(\psi^{-1} \circ \tau_{\psi(y)-\varphi(x)} \circ \varphi\right)$ is in $\Pi_G(M,M)$, i.e., $\Pi_G(M,M)$ is integrable.
Finally, to prove the converse we only have to construct $\omega_G(M)$ using Eq. (3.18) and repeat the above construction of a $G-$structure which inverts $\Pi^1_G(M,M)$.

Remark 3.1.16. Let $\omega_G(M)$ be a $G-$structure on M and $\overline{\omega}_{\overline{G}}(M)$ be a $\overline{G}-$structure on M such that $\mathcal{G}(\omega_G(M)) = \mathcal{G}\left(\overline{\omega}_{\overline{G}}(M)\right)$, i.e., for all $j^1_{0,x}\phi, j^1_{0,y}\theta \in \omega_G(M)$, there exist $j^1_{0,x}\overline{\phi}, j^1_{0,y}\overline{\theta} \in \overline{\omega}_{\overline{G}}(M)$ such that

$$j^1_{x,y}\left(\theta \circ \phi^{-1}\right) = j^1_{x,y}\left(\overline{\theta} \circ \overline{\phi}^{-1}\right).$$

Then,

$$j^1_{0,0}\left(\theta^{-1} \circ \overline{\theta}\right) = j^1_{0,0}\left(\phi^{-1} \circ \overline{\phi}\right).$$

So, denoting by $Z_0 = j^1_{0,0}\left(\phi^{-1} \circ \overline{\phi}\right)$, we have

$$\omega_G(M) \cdot Z_0 = \overline{\omega}_{\overline{G}}(M), \tag{3.19}$$

In fact, for $j^1_{0,x}\psi \in \omega_G(M)$ we have that

$$\left(j^1_{0,x}\psi\right) \cdot Z_0 = j^1_{x,x}\left(\psi \circ \phi^{-1}\right) \cdot j^1_{0,x}\overline{\phi} \in \overline{\omega}_{\overline{G}}(M),$$

taking into account that

$$j^1_{x,x}\left(\psi \circ \phi^{-1}\right) \in \mathcal{G}\left(\overline{\omega}_{\overline{G}}(M)\right).$$

Hence,

$$\omega_G(M)_x \cdot Z_0 \subseteq \overline{\omega}_{\overline{G}}(M)_x.$$

The converse is proved in the same way and, so

$$\omega_G(M)_x \cdot Z_0 = \overline{\omega}_{\overline{G}}(M)_x. \tag{3.20}$$

Finally, in general, if Eq. (3.20) is satisfied for one point it is easy to prove that

$$\omega_G(M) \cdot Z_0 = \overline{\omega}_{\overline{G}}(M).$$

Note that this implies that the isotropy groups are conjugate, namely

$$\overline{G} = Z_0 \cdot G \cdot Z_0^{-1}.$$

This kind of $G-$structures are called *conjugated $G-$structures.*

Conversely, for all $\omega_G(M), \overline{\omega}_{\overline{G}}(M)$, conjugated $G-$structures, we have

$$\mathcal{G}(\omega_G(M)) = \mathcal{G}\left(\overline{\omega}_{\overline{G}}(M)\right).$$

Using this, if $\mathcal{G}(\omega_G(M))$ is integrable, then $\omega_G(M)$ is integrable too.

Let $\mathcal{B}$ be a smoothly uniform body. Using the above results, the G_0-structure $\omega_{G_0}(\mathcal{B})$ is integrable if, and only if, $\mathcal{G}\left(\omega_{G_0}(\mathcal{B})\right)$ is integrable. Furthermore, it is lear by construction that

$$\mathcal{G}\left(\omega_{G_0}(\mathcal{B})\right) = \Omega(\mathcal{B}).$$

Therefore, using propositions 3.1.5 and 2.1.14, we have effectively proved that both definitions are equivalent.

Example

We will use proposition 3.1.10 to work with an example. The example is based on the model of a so-called *simple liquid crystal.* These simple materials were introduced by Coleman [14] and Wang [91].

Let $\mathcal{B}$ be a simple body (we will assume that $\mathcal{B}$ is an open subset of $\mathbb{R}^3$ by taking the image by the reference configuration ϕ_0) with a mechanical response $W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ such that for all $h = j^1_{X,Y}\phi \in \Pi^1(\mathcal{B},\mathcal{B})$ we have

$$W(h) = \widehat{W}(r(h), J(h)),$$

where, denoting by F the associated matrix to $j^1_{X,Y}\phi$ (with respect to the canonical basis of $\mathcal{B}$),

- $r\left(j^1_{X,Y}\phi\right) = g(Y)\left(T_X\phi(e(X)), T_X\phi(e(X))\right)$
- $J\left(j^1_{X,Y}\phi\right) = det(F)$

with $e \in \mathfrak{X}(\mathcal{B})$ a vector field which is not zero at any point and g a Riemannian metric on $\mathbb{R}^3$. Notice that the tangent bundle $T\mathcal{B}$ is canonically isomorphic to $\mathcal{B} \times \mathbb{R}^3$. So, for each $Y \in \mathcal{B}$, $g(Y)$ can be seen as a inner product on $\mathbb{R}^3$. Then, the expression of r turns into the following,

$$r\left(j^1_{X,Y}\phi\right) = g(Y)\left(F \cdot \left(e^I(X)\right), F \cdot \left(e^I(X)\right)\right),$$

where, by using the canonical isomorphism $T\mathcal{B} \to \mathcal{B} \times \mathbb{R}^3$, for all $X \in \mathcal{B}$ $e(X) = \left(X, e^I(X)\right)$. We will use both expressions with the same notation. Now, we want to study the condition which characterizes the material algebroid: A left-invariant vector field $\Theta \in \mathfrak{X}_L\left(\Pi^1(\mathcal{B},\mathcal{B})\right)$ restricts to a section of $A\Omega(\mathcal{B})$ if, and only if,

$$\Theta(W) = 0.$$

So, we should study TW over left-invariant vector fields. Let $\Theta \in \mathfrak{X}_L\left(\Pi^1(\mathcal{B},\mathcal{B})\right)$ be a left-invariant vector field and consider the canonical local system of coordinates $\left(X^I\right)$ in $\mathbb{R}^3$ restricted to $\mathcal{B}$. Notice that, in fact, $\left(X^I\right)$ is the system of coordinates generated by the reference configuration ϕ_0. We will denote by $\left(X^I, Y^J, F_I^J\right)$ the induced local coordinates of $\left(X^I\right)$ in $\Pi^1(\mathcal{B}, \mathcal{B})$. The local expression of Θ will be denoted as follows,

$$\Theta\left(X^I, Y^J, F_I^J\right) = \left(\left(X^I, Y^J, F_I^J\right), \delta X^I, 0, F_L^J \delta P_I^L\right).$$

Now, we will begin given the derivatives of r and J. For each $A \in gl(3, \mathbb{R})$ and $v \in \mathbb{R}^3$ we have that,

(i) $$\frac{\partial r}{\partial X_{|j^1_{X,Y}}}(v) = 2g(Y)\left(T_X\phi\left(e(X)\right), T_X\phi\left(\frac{\partial e}{\partial X_{|X}}(v)\right)\right)$$

(ii) $$\frac{\partial r}{\partial F_{|j^1_{X,Y}}}(A) = 2g(Y)\left(F \cdot \left(e^I(X)\right), A \cdot \left(e^I(X)\right)\right)$$

(iii) $$\frac{\partial J}{\partial F_{|j^1_{X,Y}}}(A) = det(F)\, Tr\left(F^{-1} \cdot A\right)$$

Here F is the Jacobian matrix of ϕ at X and $\frac{\partial e}{\partial X_{|X}}(v)$ is the vector at X such that

$$\frac{\partial e}{\partial X_{|X}}(v) = \left(X, \frac{\partial e^I}{\partial X^L_{|X}} v^L\right).$$

Hence, Θ restricts to a section of the material algebroid $A\Omega(\mathcal{B})$ if, and only if,

$$\begin{aligned}
0 \;=\;& 2\frac{\partial \widehat{W}}{\partial r_{|j^1_{X,Y}}} g(Y)\left(T_X\phi\left(e(X)\right), T_X\phi\left(\frac{\partial e}{\partial X_{|X}}\left(\delta X^i(X)\right)\right)\right) \\
& +2\frac{\partial \widehat{W}}{\partial r_{|j^1_{X,Y}}} g(Y)\left(F \cdot \left(e^I(X)\right), F_l^j \delta P_i^l(X) \cdot \left(e^i(X)\right)\right) + \\
& +det(F)\frac{\partial \widehat{W}}{\partial J_{|j^1_{X,Y}}} Tr\left(\delta P_i^j(X)\right),
\end{aligned}$$

for all $j^1_{X,Y}\phi \in \Pi^1(\mathcal{B},\mathcal{B})$. So, a sufficient but not necessary condition would be

(1) $Tr\left(\delta P^J_I(X)\right) = 0$

(2) Denoting by $\mathcal{L}_X = \frac{\partial e}{\partial X_{|X}}\left(\delta X^I(X)\right) + \delta P^J_I(X)\cdot\left(e^I(X)\right)$, then,

$$g(Y)\left(F\cdot\left(e^I(X)\right), F^R_M\cdot\mathcal{L}_X\right) = 0$$

By using that g is non-degenerate and $e(X)$ is non-zero, we turn these conditions into the following

$$\textbf{(1)'}\qquad \delta P^I_I = 0.$$

$$\textbf{(2)'}\qquad \frac{\partial e^J}{\partial X^L}\delta X^L + \delta P^J_L e^L = 0,\ \forall J,$$

where e^J are the coordinates of e respect to $\left(X^J\right)$.
Let us now study the uniformity of the material. By using proposition 2.3.48 $\mathcal{B}$ is uniform if, and only if, the material algebroid of $\mathcal{B}$ is transitive. Let $V_X = \left(X, V^I\right)$ be a vector at $X\in\mathcal{B}$. Then, we should find a (local) left-invariant vector field Θ such that

- $\Theta(W) = 0.$
- $T_{\epsilon(X)}\alpha\left(\Theta\left(\epsilon(X)\right)\right) = V_X,$

where ϵ and α are the identities map and the source map of the material groupoid respectively.
Let us fix the local expression of Θ as follows,

$$\Theta\left(X^I, Y^J, F^J_I\right) = \left(\left(X^I, Y^J, F^J_I\right), \delta X^I, 0, F^J_L\delta P^L_I\right).$$

Then,

$$T_{\epsilon(X)}\alpha\left(\Theta\left(\epsilon(X)\right)\right) = \left(X^I(X), \delta X^I(X)\right).$$

So, it should satisfy that,

$$\delta X^I(X) = V^I,\ \forall i.$$

By taking into account identities **(1)'** and **(2)'**, it is enough to find a family of (local) maps A_i^j from the body to the space of matrices satisfying that

$$\textbf{(1)''} \qquad A_I^I = 0.$$

$$\textbf{(2)''} \qquad \frac{\partial e^J}{\partial X^L} V^L = -A_L^J e^L, \ \forall J,$$

It is just an easy exercise to prove that there are infinite solutions A_I^J of the equations **(1)"** and **(2)"** and, hence, $\mathcal{B}$ is (smoothly) uniform.

From now on, we will assume that $\widehat{W}$ is an immersion. In that way, **(1)'** and **(2)'** are also necessary conditions.

Next, we will study the condition of (local) homogeneity. As we know (proposition 3.1.10) $\mathcal{B}$ is (locally) homogeneous if, and only if, there exists a local system of coordinates $\left(x^i\right)$ such that the local section of $\sharp$,

$$\Delta\left(x^i, \frac{\partial}{\partial x^i}\right) = \left(x^i, \frac{\partial}{\partial x^i}, 0\right),$$

takes values in the material algebroid $A\Omega\left(\mathcal{B}\right)$. Equivalently,

$$\frac{\partial W}{\partial x^i} = 0, \ \forall i. \tag{3.21}$$

So, let us study this equality. Notice that,

$$\frac{\partial W}{\partial x^i} = \frac{\partial \widehat{W}}{\partial r}\frac{\partial r}{\partial x^i} + \frac{\partial \widehat{W}}{\partial J}\frac{\partial J}{\partial x^i}.$$

Thus, by using that $\widehat{W}$ is an immersion, $\left(x^i\right)$ are homogeneous coordinates if, and only if,

$$\textbf{(1)'''} \qquad \frac{\partial r}{\partial x^i} = 0, \ \forall i.$$

$$\textbf{(2)'''} \qquad \frac{\partial J}{\partial x^i} = 0, \ \forall i.$$

Observe that the form of $\widehat{W}$ is not important to evaluate the (local) homogeneity of $\mathcal{B}$ as long as $\widehat{W}$ is an immersion.
Let $\left(x^i\right)$ be a system of homogeneous coordinates on $\mathcal{B}$. Then, for each $j^1_{X,Y}\phi \in \Pi^1\left(\mathcal{B},\mathcal{B}\right)$

$$\begin{aligned} r\left(j^1_{X,Y}\phi\right) &= g\left(Y\right)\left(T_X\phi\left(e\left(X\right)\right), T_X\phi\left(e\left(X\right)\right)\right) \\ &= g\left(Y\right)\left(T_X\phi\left(e^i\left(X\right)\frac{\partial}{\partial x^i_{|X}}\right), T_X\phi\left(e^j\left(X\right)\frac{\partial}{\partial x^j_{|X}}\right)\right) \\ &= e^i\left(X\right)e^j\left(X\right)\frac{\partial \phi^k}{\partial x^i_{|X}}\frac{\partial \phi^l}{\partial x^j_{|X}}g_{kl}\left(Y\right), \end{aligned}$$

where, in this case, e^j are the coordinates of e respect to $\left(x^i\right)$. So, considering the induced coordinates $\left(x^i, y^j, y^j_i\right)$ of $\left(x^i\right)$ on $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$ we have that

$$r \circ \left(x^i, y^j, y^j_i\right)^{-1}\left(\tilde{X}, \tilde{Y}, \tilde{F}\right) = e^i\left(X\right)e^j\left(X\right)\tilde{F}^k_i\tilde{F}^l_j g_{kl}\left(Y\right),$$

for all $\left(\tilde{X}, \tilde{Y}, \tilde{F}\right)$. In this way,

$$\frac{\partial r}{\partial x^k_{|j^1_{X,Y}}} = 2\frac{\partial e^i}{\partial x^k_{|X}}e^j\left(X\right)\tilde{F}^k_i\tilde{F}^l_j g_{kl}\left(Y\right).$$

Hence, by using the non-degeneracy of g we have that $\dfrac{\partial r}{\partial x^k} = 0$ if, and only if,

$$\frac{\partial e^i}{\partial x^k} = 0, \ \forall i. \tag{3.22}$$

With this, **(1)”** is satisfied if, and only if, the vector field e is constant respect to $\left(x^i\right)$, i.e.,

$$e = \lambda^i \frac{\partial}{\partial x^i}, \ \lambda^i \equiv Const. \tag{3.23}$$

Next, we will study condition **(2)”**. Notice that,

$$\frac{\partial J}{\partial x^i} = \frac{\partial J}{\partial F^L_M} \frac{\partial F^L_M}{\partial x^i}.$$

Using the derivative of J (which we have shown above), we have that

$$\frac{\partial J}{\partial F^L_{M\,|\tilde{F}}} = det\left(\tilde{F}\right)\left(\tilde{F}^{-1}\right)^L_M.$$

Then, **(2)”** is satisfied if, and only if,

$$\frac{\partial F^L_M}{\partial x^i} = 0, \ \forall i, L, M. \tag{3.24}$$

Observe that

$$\begin{aligned} \frac{\partial F^L_M}{\partial x^k_{|j^1_{X,Y}}} &= \frac{\partial F^L_M \circ \left(x^i, y^j, y^j_i\right)^{-1}}{\partial X^K_{|\left(\tilde{X},\tilde{Y},\tilde{F}\right)}} \\ &= \frac{\partial}{\partial X^K_{\left(\tilde{X},\tilde{Y},\tilde{F}\right)}}\left(\frac{\partial X^L \circ \left(y^j\right)^{-1}}{\partial X^K_{|\tilde{Y}}} \cdot \tilde{F}^K_R \cdot \left[\frac{\partial X^R \circ \left(x^i\right)^{-1}}{\partial X^M_{|\tilde{X}}}\right]^{-1}\right). \end{aligned}$$

i.e.,

$$\frac{\partial F^L_M}{\partial x^k_{|j^1_{X,Y}}} = 0,$$

if, and only if,

$$\frac{\partial}{\partial X^K_{|\tilde{X}}}\left(\frac{\partial X^M \circ \left(x^i\right)^{-1}}{\partial X^I_{|\tilde{X}}}\right) = 0.$$

So, **(2)”** is tantamount to,

$$\frac{\partial X^M}{\partial x^i} \equiv Const, \ \forall i, M.$$

This fact implies that,

$$e\left(X^M\right) \equiv Const,\ \forall m.$$

i.e.,

$$e = \mu^I \frac{\partial}{\partial X^I},\ \mu^I \equiv Const.$$

Notice that, by using Eq. (3.23), this implies, indeed, that the canonical basis (and hence the reference configuration of $\mathcal{B}$) is a (global) system of homogeneous coordinates on $\mathcal{B}$. So, we extract the following conclusions

(a) $\mathcal{B}$ is (locally) homogeneous if, and only if, the vector field e is constant respect to the canonical basis of $\mathbb{R}^3$.

(b) The homogeneity of $\mathcal{B}$ implies that the reference coordinates are homogeneous coordinates.

(c) $\mathcal{B}$ is locally homogeneous if, and only if, $\mathcal{B}$ is global homogeneous.

3.2 Cosserat media

The content of this section may be found summarized in the published article [52] which is included in the collection of articles in which this thesis consists. We present a similar development to the previous section for simple media.

Uniformity and homogeneity

Analogously as we did for simple materials in **Prelude** 2.3, we should specify how the structure of Cosserat media change by using groupoids. Let $F\mathcal{B}$ be a Cosserat media with reference configuration Φ_0 and mechanical response $W : \mathcal{B} \times Gl\left(12, \mathbb{R}\right) \rightarrow V$. The rule of change of configurations (2.19) permits us to define W not only in configurations but in the second-order non-holonomic groupoid $\tilde{J}^1\left(F\mathcal{B}\right)$ over the macromedium $\mathcal{B}$ (see example 2.2.23). In fact, let be $j^1_{X,Y}\Psi \in \tilde{J}\left(F\mathcal{B}\right)$, then we define

$$W\left(j^1_{X,Y}\Psi\right) = W\left(X, \overline{F}\right),$$

where $\overline{F}$ is the associated matrix to the class of 1−jets at $\phi_0(X)$ of $\Phi_0 \circ \Phi \circ \Phi_0^{-1}$. So, W may be equivalently described as a differentiable map $W : \tilde{J}^1(F\mathcal{B}) \to V$ from the groupoid of 1−jets $\tilde{J}^1(F\mathcal{B})$ to the vector space V which does not depend on the image point of the class of 1−jets of $\tilde{J}^1(F\mathcal{B})$.
Obviously, we may define W over $\tilde{J}^1\left(F\mathcal{B}, F\mathbb{R}^3\right)$, which could be seen as an open subset of $\tilde{J}^1\left(F\mathbb{R}^3\right)$ given by the class of 1−jets of local isomorphisms from $F\mathcal{B}$ to $F\mathbb{R}^3$.
Then, condition of being materially isomorphic is rewritten as follows: *Two material particles X and Y are materially isomorphic if, and only if, there exists a local principal bundle isomorphism over the identity map on $Gl(3,\mathbb{R})$, Ψ, from $F\mathcal{U} \subseteq F\mathcal{B}$ with $X \in \mathcal{U}$ to $F\mathcal{V} \subseteq F\mathcal{B}$ with $Y \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are open neighbourhood of M, such that $\psi(X) = Y$ and*

$$W\left(j^1_{Y,\kappa(Y)}\tilde{\kappa} \circ j^1_{X,Y}\Psi\right) = W\left(j^1_{Y,\kappa(Y)}\tilde{\kappa}\right), \tag{3.25}$$

for all $j^1_{Y,\kappa(Y)}\tilde{\kappa} \in \tilde{J}^1(F\mathcal{B})$.
For any two points X and Y, we will denote by $\overline{G}(X,Y)$ the collection of all classes of 1−jets $j^1_{X,Y}\Psi$ which satisfy Eq. (3.25). Thus, analogously to simple media, the set $\overline{\Omega}(\mathcal{B}) = \cup_{X,Y\in\mathcal{B}}\overline{G}(X,Y)$ can be considered as a groupoid over $\mathcal{B}$ which is, indeed, a subgroupoid of the second-order non-holonomic groupoid $\tilde{J}^1(F\mathcal{B})$. $\overline{\Omega}(\mathcal{B})$ will be called *non-holonomic material groupoid of second order associated to a Cosserat continuum $F\mathcal{B}$*.
So, as an abuse of notation, we will denote the structure maps of $\overline{\Omega}(\mathcal{B})$ like the structure maps of $\tilde{J}^1(F\mathcal{B})$. We will also denote $\overline{\alpha}^{-1}(X)$ (resp. $\overline{\beta}^{-1}(X)$) by $\overline{\Omega}_X(\mathcal{B})$ (resp. $\overline{\Omega}^X(\mathcal{B})$). Notice that, the material symmetry group $\overline{G}(X)$ at a body point $X \in \mathcal{B}$ is just the isotropy group of $\overline{\Omega}(\mathcal{B})$ at X.

Proposition 3.2.1. *Let $F\mathcal{B}$ be a Cosserat medium. $F\mathcal{B}$ is uniform if, and only if, $\overline{\Omega}(\mathcal{B})$ is a transitive subgroupoid of $\tilde{J}^1(F\mathcal{B})$.*

As a consequence of the continuity of W we have that, for all $X \in \mathcal{B}$, $\overline{G}(X)$ is a closed subgroup of $\tilde{J}^1(F\mathcal{B})^X_X$.

Proposition 3.2.2. *Let $\mathcal{B}$ be a simple body. Then, for all $X \in \mathcal{B}$ the symmetry group $\overline{G}(X)$ is a Lie subgroup of $\tilde{J}^1(F\mathcal{B})^X_X$.*

Next, by composing appropriately with the reference configuration, smooth uniformity (definition 2.1.20) may be rewritten as follows.

Proposition 3.2.3. A Cosserat medium $F\mathcal{B}$ is smoothly uniform if, and only if, for each point $X \in \mathcal{B}$ there is an neighbourhood $\mathcal{U}$ around X such that for all $Y \in \mathcal{U}$ and $j^1_{Y,X}\Psi \in \overline{\Omega}(\mathcal{B})$ there exists a local section $\overline{\mathcal{P}}$ of

$$\overline{\alpha}_X : \overline{\Omega}^X(\mathcal{B}) \to \mathcal{B},$$

from $\overline{\epsilon}(X)$ to $j^1_{Y,X}\Psi$, where $\overline{\alpha}_X$ is the restriction of the source map $\overline{\alpha}$ to the beta fibre $\overline{\Omega}^X(\mathcal{B})$ at X.

One more time, (local) sections of $\overline{\alpha}_X$ (resp. $\overline{\beta}_X$) will be called *left fields of material isomorphism at* X (resp. *right fields of material isomorphism at* X) because they are in a bijective correspondence with the left (resp. right) fields of material isomorphisms defined in 2.1 via composition with the reference configuration Φ_0.
Therefore, $\mathcal{B}$ is smoothly uniform if, and only if, for each two points $X, Y \in \mathcal{B}$ there are two open subsets $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$ around X and Y respectively and $\overline{\mathcal{P}} : \mathcal{U} \times \mathcal{V} \to \overline{\Omega}(\mathcal{B}) \subseteq \tilde{J}^1(F\mathcal{B})$, a differentiable section of the anchor map $\left(\overline{\alpha}, \overline{\beta}\right)$. When $X = Y$ it is easy to realize that we can assume $\mathcal{U} = \mathcal{V}$ and $\mathcal{P}$ is a morphism of groupoids over the identity map, i.e.,

$$\overline{\mathcal{P}}(Z,T) = \overline{\mathcal{P}}(R,T)\,\overline{\mathcal{P}}(Z,R)\,, \;\forall T, R, Z \in \mathcal{U}.$$

As in the case of simple materials, we may prove a corollary of proposition 3.2.5.

Corollary 3.2.4. *Let* $F\mathcal{B}$ *be a Cosserat medium.* $F\mathcal{B}$ *is smoothly uniform if, and only if,* $\overline{\Omega}(\mathcal{B})$ *is a transitive Lie subgroupoid of* $\tilde{J}^1(F\mathcal{B})$*.*

Proof. Repeat the proof of corollary 2.3.48. □

From now on, we will assume that the non-holonomic material groupoid of second order $\overline{\Omega}(\mathcal{B})$ is a Lie subgroupoid of $\tilde{J}^1(F\mathcal{B})$.

Let us now present our candidate for definition of (locally) homogeneous Cosserat media.

Definition 3.2.5. A Cosserat medium $\mathcal{B}$ is said to be *homogeneous* if it admits a global diffeomorphism $\tilde{\kappa}$ which induces a global section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\overline{\Omega}\left(\mathcal{B}\right)$, $\overline{\mathcal{P}}$, i.e., for each $X, Y \in \mathcal{B}$

$$\overline{\mathcal{P}}\left(X, Y\right) = j^{1}_{X,Y}\left(\tilde{\kappa}^{-1} \circ F\tau_{\kappa(Y)-\kappa(X)} \circ \tilde{\kappa}\right),$$

where $\tau_{\kappa(Y)-\kappa(X)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\kappa\left(Y\right) - \kappa\left(X\right)$. $\mathcal{B}$ is said to be *locally homogeneous* if there exists a covering of $\mathcal{B}$ by homogeneous open sets.

Those (local) configurations which generate (local) deformations satisfying the definition of homogeneity are called *homogeneous configurations* or *homogeneous coordinates.*

Now, suppose that $\mathcal{B}$ is homogeneous. Then, if we take the global homogeneous coordinates $\left(x^i\right)$ on $\mathcal{B}$, we deduce that $\overline{\mathcal{P}}$ is expressed by

$$\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, P^j_i\right), \delta^j_i, \frac{\partial P^j_i}{\partial x^k} + \frac{\partial P^j_i}{\partial y^k}\right), \tag{3.26}$$

If $\mathcal{B}$ is locally homogeneous we can cover $\mathcal{B}$ by local coordinate systems $\left(x^i\right)$ which generate (local) sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\overline{\Omega}\left(\mathcal{B}\right)$ satisfying Eq. (B.0.17).

Non-holonomic second-order derivations

Firt at all, we will use the Lie algebroid isomorphism $\mathcal{D} : A\Pi^1\left(M, M\right) \to \mathfrak{D}\left(TM\right)$ (see section 2.3) to give another interpretation of the second-order non-holonomic algebroid. As a first approximation to the second-order non-holonomic case we will restrict the mentioned isomorphism to a particular case.

Consider the $1-$jets groupoid on FM, $\Pi^1\left(FM, FM\right) \rightrightarrows FM$ and $J^1\left(FM\right) \rightrightarrows FM$ the Lie subgroupoid of all $1-$jets of local automorphisms on FM.
Let $\left(x^i\right)$ and $\left(y^j\right)$ be local coordinate systems over two open sets $U, V \subseteq M$; then, the induced coordinate systems over FM are denoted by

$$FU : \left(x^i, x^i_j\right)$$

$$FV : \left(y^j, y_i^j\right).$$

Hence, (see Eq. (2.26)) we can construct induced coordinates over $J^1(FM)$,

$$J^1(FU, FV) : \left(\left(x^i, x_j^i\right), \left(y^j, y_i^j\right), y_{,i}^j, y_{i,k}^j\right).$$

So, we can consider its associated Lie algebroid $AJ^1(FM)$ as a reduced subalgebroid of the 1−jets algebroid $A\Pi^1(FM, FM)$ and, hence, its induced coordinates will be

$$AJ^1(FU) : \left(\left(x^i, x_j^i\right), \left(v^j, v_i^j\right), v_{,i}^j, 0, v_{i,k}^j, v_{i,kl}^j\right) \cong \tag{3.27}$$

$$\cong \left(\left(x^i, x_j^i\right), \left(v^j, v_i^j\right), v_{,i}^j, v_{i,k}^j\right),$$

where,

$$v_{i,kl}^j = \left(v_m^j \left(x^{-1}\right)_k^m\right) \delta_l^i.$$

In this way, we can restrict the Lie algebroids isomorphism $\mathcal{D} : A\Pi^1(FM, FM) \to \mathfrak{D}(TFM)$ to get another isomorphism between Lie algebroids, $\mathcal{D}^1 : AJ^1(FM) \to \mathfrak{D}^1(FM)$, where $\mathfrak{D}^1(FM)$ is the resulting Lie algebroid from the restriction of the isomorphism.
Let Λ be a section of $AJ^1(FM)$ such that (locally)

$$\Lambda\left(x^i, x_j^i\right) = \left(\left(x^i, x_j^i\right), \left(\Lambda^j, \Lambda_i^j\right), \Lambda_{,i}^j, \Lambda_{i,k}^j\right) \tag{3.28}$$

Then, the associated derivation is characterized by the following identities

(i) $D^{\Lambda} \dfrac{\partial}{\partial x^i} = \Lambda_{,i}^j \dfrac{\partial}{\partial x^j} + \Lambda_{k,i}^j \dfrac{\partial}{\partial x_k^j}.$

(ii) $D^{\Lambda} \dfrac{\partial}{\partial x_j^i} = \left(\Lambda_m^k \left(x^{-1}\right)_i^m\right) \dfrac{\partial}{\partial x_j^k}.$

So, conditions (i) and (ii) characterize the sections of Lie algebroid $\mathfrak{D}^1(FM)$. This space will be denoted by $Der^1(FM)$.

Remark 3.2.6. We can characterize $Der^1(FM)$ in the following way. Let $\{(\Phi_t, \psi_t)\}$ be the flow of D^Λ. Then,

$$D^\Lambda \overline{\chi} = \frac{\partial \Phi_t^* \overline{\chi}}{\partial t_{|0}}, \; \forall \overline{\chi} \in \mathfrak{X}(FM).$$

Hence, by uniqueness,

$$\left(\varphi_t^\Lambda\right)^* (\overline{\chi}) = \Phi_t^* \overline{\chi},$$

where φ_t^Λ defines the local flow of the left-invariant vector field associated to Λ. Thus, we can say that $D^\Lambda \in Der^1(FM)$ if, and only if, its flow consists of tangent maps of automorphisms of frame bundles (over the identity map).

Finally, we will work with the second-order non-holonomic algebroid $A\tilde{J}^1(FM)$ (see example 2.3.32). As we know (see examples 2.3.12 and 2.3.29), $A\tilde{J}^1(FM)$ can be seen as the quotient Lie algebroid by the induced action of Φ 2.28 over $AJ^1(FM)$.
In this way we can consider a relation in $\mathfrak{D}^1(FM)$ given by the restriction of the isomorphism $\mathcal{D}$, $\mathcal{D}^1 : AJ^1(FM) \to \mathfrak{D}^1(FM)$, and the relation in $AJ^1(FM)$, i.e.,

$$\mathcal{D}^1(a) \sim \mathcal{D}^1(b) \Leftrightarrow a \sim b, \; \forall a, b \in AJ^1(FM). \tag{3.29}$$

The new quotience space is denoted by $\tilde{\mathfrak{D}}^1(FM)$ and it is obvious that this space inherit the Lie algebroid structure from $A\tilde{J}^1(FM)$. In fact, considering

$$\tilde{\mathcal{D}} : A\tilde{J}^1(FM) \to \tilde{\mathfrak{D}}^1(FM), \tag{3.30}$$

the map which commutes with the projections, the Lie algebroid structure over $\tilde{\mathfrak{D}}^1(FM)$ is the unique Lie algebroid structure such that $\tilde{\mathcal{D}}$ is a Lie algebroid isomorphism over the identity map on M. This Lie algebroid will be called *second-order non-holonomic algebroid of derivations on TM.*
Let (x^i) be a local coordinate system on an open set $U \subseteq M$. Using Eq. (2.50) we can construct induced local coordinates over $A\tilde{J}^1(FM)$ as follows:

$$A\tilde{J}^1(FU) \cong \left(x^i, v^i, v^i_j, v^i_{,j}, v^i_{j,k}\right).$$

Thus, the non-holonomic second-order derivation algebroid is characterized by the following equalities:

(i) $D\dfrac{\partial}{\partial x^i} = f^j_{,i}\dfrac{\partial}{\partial x^j} + f^j_{k,i}\dfrac{\partial}{\partial x^j_k}$

(ii) $D\dfrac{\partial}{\partial x^i_j} = f^k_i\dfrac{\partial}{\partial x^k_j}$

where the local fuctions $f^j_{,i}, f^j_{k,i}$ and f^k_i do not depend on x^i_j.
Observe that, we could restrict $\mathcal{D}$ to $Aj^1(FM)$ and we obtain a Lie subalgebroid of $\mathfrak{D}^1(FM)$ which is denoted by $\mathfrak{d}^1(FM)$. Proceeding in the same way as in the case of $\tilde{J}^1(FM)$, we obtain a reduced Lie subalgebroid of $\tilde{\mathfrak{D}}^1(FM)$. This Lie algebroid is denoted by $\tilde{\mathfrak{d}}^1(FM)$ and it is called *second-order holonomic algebroid of derivations on TM.* Obviously, this subalgebroid is isomorphic to holonomic algebroid of second order $A\tilde{j}^1(FM)$ by restricting $\tilde{\mathcal{D}}$.

The second-order holonomic algebroid of derivations on TM is characterized by the above equalities satisfying these two additional conditions

$$f^j_i = f^j_{,i} \quad ; \quad f^j_{i,k} = f^j_{k,i}.$$

Remark 3.2.7. Denote by $\Gamma\left(AJ^1(FM)\right)^G$ the set of $(A\Phi, R)$ −invariant sections of $AJ^1(FM)$, i.e., for all $\Lambda \in \Gamma\left(AJ^1(FM)\right)^G$ and $g \in Gl(n,\mathbb{R})$, the diagram

$$\begin{array}{ccc} FM & \xrightarrow{\quad\Lambda\quad} & AJ^1(FM) \\ R_g \downarrow & & \downarrow A\Phi_g \\ FM & \xrightarrow{\quad\Lambda\quad} & AJ^1(FM) \end{array}$$

is commutative, namely

$$\Lambda(\overline{x}\cdot g) = T_{\epsilon(\overline{x})}\Phi^{\overline{x}}_g(\Lambda(\overline{x})),\ \forall \overline{x} \in FM,\ \forall g \in Gl(n,\mathbb{R}),$$

where $\Phi_g^{\overline{x}} : \beta^{-1}(\overline{x}) \to \beta^{-1}(\overline{x} \cdot g)$ is the restriction of Φ_g to $\beta^{-1}(\overline{x})$.

As we have seen in example 2.3.12, the space of $(A\Phi, R)$-invariant sections of $AJ^1(FM)$ is isomorphic to $\Gamma\left(A\tilde{J}^1(FM)\right)$.
Next, take $\Lambda \in \Gamma\left(AJ^1(FM)\right)^G$. Then, it satisfies that

$$\varphi_t^{\Lambda}(\epsilon(\overline{x})) \circ R_g = \Phi_g \circ \varphi_t^{\Lambda}(\epsilon(\overline{x})), \; \forall g \in Gl(n, \mathbb{R}). \tag{3.31}$$

Now, let $\left(\varphi_t^{\Lambda}\right)^* : \mathfrak{X}(FM) \to \mathfrak{X}(FM)$ be the induced linear map over $\mathfrak{X}(FM)$ of φ_t^{Λ}. Let be $\overline{\chi} \in \mathfrak{X}(FM)$; then, we have that for all $g \in Gl(n, \mathbb{R})$

$$\begin{aligned}
& \{\left(\varphi_t^{\Lambda}\right)^* \circ R_g^*(\overline{\chi})\}(\overline{x}) = \\
= \; & \varphi_t^{\Lambda}(\epsilon(\overline{x}))\left(T_{\left(\alpha\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}))\right)\right)\cdot g^{-1}} R_g\left(\overline{\chi}\left(\alpha\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}))\right)\cdot g^{-1}\right)\right)\right) \\
= \; & T_{\overline{x}\cdot g^{-1}} R_g\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}\cdot g^{-1}))\left(\overline{\chi}\left(\alpha\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}))\right)\cdot g^{-1}\right)\right)\right) \\
= \; & T_{\overline{x}\cdot g^{-1}} R_g\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}\cdot g^{-1}))\left(\overline{\chi}\left(\alpha\left(\varphi_t^{\Lambda}(\epsilon(\overline{x}\cdot g^{-1}))\right)\right)\right)\right) \\
= \; & \{R_g^* \circ \left(\varphi_t^{\Lambda}\right)^*(\overline{\chi})\}(\overline{x})
\end{aligned}$$

Notice that, here R_g^* denotes the pullback of $R_g : FM \to FM$. Conversely, suppose that $\Lambda \in \Gamma\left(AJ^1(FM)\right)$ satisfies the above equality. Then, in a similar way, we can prove that Λ is $(A\Phi, R)$ $-$invariant. Taking derivatives in this equality, we have that it is equivalent to

$$D^{\Lambda} \circ R_g^* = R_g^* \circ D^{\Lambda}. \tag{3.32}$$

So, we have proved that the space $\tilde{\mathfrak{D}}^1(FM)$ can be seen as the derivations in $\mathfrak{D}^1(FM)$ which commute with R_g^* for all $g \in Gl(n, \mathbb{R})$. These derivations are called *non-holonomic derivations of second order.*
Analogously, the second-order holonomic algebroid of derivations $\tilde{\mathfrak{d}}^1(FM)$ can be seen as the derivations in $\mathfrak{d}^1(FM)$ which commute with R_g^* for all $g \in Gl(n, \mathbb{R})$.

Observe that, if D^{Λ} satisfies Eq. (3.32) then, its base vector field $\Lambda^{\sharp} \in \mathfrak{X}(FM)$ is right-invariant, .i.e.

$$R_g^* \Lambda^{\sharp} = \Lambda^{\sharp}, \ \forall g \in Gl(n, \mathbb{R}),$$

or, equivalently,

$$T_{\overline{z}} R_g \left(\Lambda^{\sharp}(\overline{z}) \right) = \Lambda^{\sharp}(\overline{z} \cdot g),$$

for all $\overline{z} \in FM$. Thus, $\Lambda^{\sharp}$ is π_M-related with a (unique) vector field on M.
Let (x^i) be local coordinates on M and $\overline{\Lambda}$ be a section of $A\tilde{J}^1(FM)$ which satisfies that

$$\overline{\Lambda}(x^i) = \left(x^i, \Lambda^i, \Lambda^i_j, \Lambda^i_{,j}, \Lambda^i_{j,k} \right).$$

Then, its associated $(A\Phi, R)$-invariant section Λ of $AJ^1(FM)$ is given by

$$\Lambda\left(x^i, x^i_j\right) = \left(x^i, x^i_j, \Lambda^i, \Lambda^i_l x^l_j, \Lambda^i_{,j}, \Lambda^i_{j,k} \right).$$

Hence,

$$\Lambda^{\sharp}\left(x^i, x^i_j\right) = \left(x^i, x^i_j, \Lambda^i, \Lambda^i_l x^l_j \right),$$

and its ρ_M-related vector field on M is

$$\overline{\Lambda}^{\overline{\sharp}}\left(x^i\right) = \left(x^i, \Lambda^i\right).$$

Finally, suppose that $\overline{\Delta}$ is a linear section of $\overline{\sharp}$. Then, we can consider the map $\overline{\Delta} : \mathfrak{X}(M) \to \Gamma\left(AJ^1(FM)\right)^G$ such that for each vector field $\chi \in \mathfrak{X}(M)$, $\overline{\Delta}(\chi)$ is the associated $(A\Phi, R)$-invariant section of $AJ^1(FM)$ to $\overline{\Delta}(\chi)$.
Then, for all $f \in \mathcal{C}^{\infty}(M)$ and $\chi \in \mathfrak{X}(M)$, we have

$$\overline{\Delta}(f\chi) = (f \circ \rho_M)\, \overline{\Delta}(\chi).$$

Hence, considering the associated derivation to $\overline{\Delta}(\chi)$ we obtain the following map

$$\nabla^{\overline{\Delta}} : \mathfrak{X}(M) \times \mathfrak{X}(FM) \to \mathfrak{X}(FM), \tag{3.33}$$

which satisfies

(i) For all $f \in \mathcal{C}^{\infty}(M)$, $\chi \in \mathfrak{X}(M)$ and $\overline{\varsigma} \in \mathfrak{X}(FM)$ we have

$$\nabla^{\overline{\Delta}}_{f\chi}\overline{\varsigma} = (f \circ \rho_M)\, \nabla^{\overline{\Delta}}_{\chi}\overline{\varsigma}.$$

(ii) For all $F \in \mathcal{C}^{\infty}(FM)$, $\chi \in \mathfrak{X}(M)$ and $\overline{\varsigma} \in \mathfrak{X}(FM)$ we have

$$\nabla^{\overline{\Delta}}_{\chi}F\overline{\varsigma} = F\nabla^{\overline{\Delta}}_{\chi}\overline{\varsigma} + \overline{\Delta}(\chi)^{\sharp}(F)\,\overline{\varsigma}.$$

(iii) For all $\chi \in \mathfrak{X}(M)$, the base vector field of $\nabla^{\overline{\Delta}}_{\chi}$ is $\overline{\Delta}(\chi)^{\sharp}$ which is ρ_M-related to χ.

(iv) For all $g \in Gl(n, \mathbb{R})$ and $\chi \in \mathfrak{X}(M)$,

$$\nabla^{\overline{\Delta}}_{\chi} \circ R^{*}_{g} = R^{*}_{g} \circ \nabla^{\overline{\Delta}}_{\chi}.$$

(v) For all $\chi \in \mathfrak{X}(M)$ the flow of $\nabla^{\overline{\Delta}}_{\chi}$ is the tangent map of an automorphism of frame bundles (over the identity map) at each fibre.

These kind of objects will be called *second-order non-holonomic covariant derivatives on M*.
Roughly speaking, the isomorphism $\tilde{\mathfrak{D}}$ gives us a way to interpret a linear section of $\overline{\sharp}$ as a map which turn a vector field $\chi \in \mathfrak{X}(M)$ into a $R^{*}_{g}-$invariant derivation over TFM with a base vector field which projects over χ. Note that, in this case, this map is not exactly a covariant derivative but it has a similar shape.

Integrability

We will now introduce the notion of *integrability* of reduced Lie subgroupoids of the second-order non-holonomic groupoid.
In order to do that, we will proceed in a similar way that in the case of $\overline{F}^2 M$. Thus, there exists a canonical Lie groupoid isomorphism over the identity on $\mathbb{R}^n$, $L : \tilde{J}^1(F\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n \times \overline{G}^2(n)$, where $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G}^2(n)$ is the trivial Lie groupoid of $\overline{G}^2(n)$ over $\mathbb{R}^n$ defined by

$$L\left(j^1_{x,y}\Psi\right) = \left(x, y, j^1_{0,0}\left(F\tau_{-y} \circ \Psi \circ F\tau_x\right)\right),\ \forall x, y \in \mathbb{R}^n,$$

where τ_{-y} and τ_x denote the translations on $\mathbb{R}^n$ by the vectors $-y$ and x respectively. Thus, if $\overline{G}$ is a Lie subgroup of $\overline{G}^2(n)$, we can transport $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G}$ by this isomorphism to obtain a reduced Lie subgroupoid of $\tilde{J}^1(F\mathbb{R}^n)$. This reduced Lie subgroupoid of $\tilde{J}^1(F\mathbb{R}^n)$ will be called *standard flat* subgroupoid of $\tilde{J}^1(F\mathbb{R}^n)$ over $\overline{G}$.

Let $U, V \subseteq M$ be two open subsets of M. We denote by $\tilde{J}^1(FU, FV)$ the open subset of $\tilde{J}^1(FM)$ defined by $\left(\overline{\alpha}, \overline{\beta}\right)^{-1}(U \times V)$. Note that if $U = V$, $\tilde{J}^1(FU, FU)$ is in fact the second-order non-holonomic groupoid over U, i.e., $\tilde{J}^1(FU, FU) = \tilde{J}^1(FU)$. Furthermore, we will think about $\tilde{J}^1(FU, FV)$ as the restriction of the Lie groupoid $\tilde{J}^1(FM)$ equipped with the restriction of the structure maps (this could not be a Lie groupoid). We will also use this notation for subgroupoids of $\tilde{J}^1(FM)$.

Next, we will introduce the notion of *integrability of a reduced Lie subgroupoid.*

Definition 3.2.8. Let $\tilde{J}^1_{\overline{G}}(FM)$ be a reduced Lie subgroupoid of $\tilde{J}^1(FM)$. $\tilde{J}^1_{\overline{G}}(FM)$ is *integrable* if it is locally isomorphic to the trivial Lie groupoid $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G}$ for some Lie subgroup $\overline{G}$ of $\overline{G}^2(n)$.

Note that $\tilde{J}^1_{\overline{G}}(FM)$ is "locally isomorphic" to $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G} \rightrightarrows \mathbb{R}^n$ if for all $x \in M$ there exist an open set $U \subseteq M$ with $x \in U$ and a local chart, $\psi_U : U \to \overline{U}$, which induces a Lie groupoid isomorphism,

$$\Psi_U : \tilde{J}^1_{\overline{G}}(FU) \to \overline{U} \times \overline{U} \times \overline{G}, \tag{3.34}$$

such that $\Psi_U = \left(\psi_U \circ \overline{\alpha}, \psi_U \circ \overline{\beta}, \overline{\Psi}_U\right)$, where for each $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{G}}(FU)$ the image $\overline{\Psi}_U\left(j^1_{x,y}\Psi\right)$ is given by

$$j^1_{0,0}\left(F\left(\tau_{-\psi_U(y)} \circ \psi_U\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right).$$

So, we can claim that $\tilde{J}^1_{\overline{G}}(FM)$ is locally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G}$ if we can cover M by local charts (ψ_U, U) that induce Lie groupoid isomorphisms from $\tilde{J}^1_{\overline{G}}(FU)$ to the restrictions of the standard flat over $\overline{G}$ to $\overline{U}$.

Remark 3.2.9. Let $\tilde{J}^1_{\overline{G}}(FM)$ be an integrable subgroupoid of $\tilde{J}^1(FM)$, i.e., locally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times \overline{G}$ with $\overline{G}$ a Lie subgroup of $\overline{G}^1(n)$. Suppose that there exists another Lie subgroup of $\overline{G}^2(n)$, $\tilde{G}$, such that $\tilde{J}^1_{\overline{G}}(FM)$ is locally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times \tilde{G}$. Then, it is easy to see that G and $\tilde{G}$ are conjugated subgroups of $\overline{G}^2(n)$, i.e., there exists a frame at 0, $\overline{g} \in \overline{G}^2(n)$, such that

$$\tilde{G} = \overline{g}^{-1} \cdot \overline{G} \cdot \overline{g}.$$

However, in this case the converse is not true.

Now, there is a special kind of reduced subgroupoids of $\tilde{J}^1(FM)$ which will play an important role in the following. A trivial reduced subgroupoid of $\tilde{J}^1(FM)$ or *parallelism of* $\tilde{J}^1(FM)$ is a reduced subgroupoid of $\tilde{J}^1(FM)$, $\tilde{J}^1_{\overline{e}}(FM)$, such that for each $x, y \in M$ there exists a unique $1-$jet $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{e}}(FM)$, or equivalently, the restriction of $\left(\overline{\alpha}, \overline{\beta}\right)$ to $\tilde{J}^1_{\overline{e}}(FM)$ is a Lie groupoid isomorphism.
So, having a trivial reduced subgroupoid $\tilde{J}^1_{\overline{e}}(FM)$ of $\tilde{J}^1(FM)$ we can consider a global section of $\left(\overline{\alpha}, \overline{\beta}\right)$, $\overline{\mathcal{P}} : M \times M \to \tilde{J}^1(FM)$, such that $\overline{\mathcal{P}}(x, y)$ is the unique $1-$jet from x to y which is in $\tilde{J}^1_{\overline{e}}(FM)$, i.e., $\overline{\mathcal{P}} = \left(\overline{\alpha}, \overline{\beta}\right)^{-1}_{|\tilde{J}^1_{\overline{e}}(FM)}$. Conversely, every global section of $\left(\overline{\alpha}, \overline{\beta}\right)$ can be seen as a parallelism of $\tilde{J}^1(FM)$ (we are understanding "section" as section in the category of Lie groupoids, i.e., Lie groupoid morphism from the pair groupoid $M \times M$ to $\tilde{J}^1(FM)$ which is a section of the morphism $\left(\overline{\alpha}, \overline{\beta}\right)$). Using this, we can also speak about *integrable sections of* $\left(\overline{\alpha}, \overline{\beta}\right)$.

Next, we will express a necessary result to interpret the integrability in another equivalent way.

Proposition 3.2.10. *Let $\tilde{J}^1_{\overline{G}}(FM)$ be a reduced Lie subgroupoid of $\tilde{J}^1(FM)$. $\tilde{J}^1_{\overline{G}}(FM)$ is integrable if, and only if, for all $x, y \in M$ there exist two open sets $U, V \subseteq M$ with $x \in U$ and $y \in V$ and two local charts $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$ which induce a diffeomorphism $\Psi_{U,V} : \tilde{J}^1_{\overline{G}}(FU, FV) \to \overline{U} \times \overline{V} \times \overline{G}$ such that for all $j^1_{x,y}\Psi$*

$$\Psi_{U,V}\left(j^1_{x,y}\Psi\right) = \left(\psi_U(x), \psi_V(y), \overline{\Psi}_{U,V}\left(j^1_{x,y}\Psi\right)\right) \tag{3.35}$$

where, $\overline{\Psi}_{U,V}\left(j^1_{x,y}\Psi\right)$ is given by

$$j^1_{0,0}\left(F\left(\tau_{-\psi_V(y)} \circ \psi_V\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right).$$

Proof. First, suppose that $\tilde{J}^1_{\overline{G}}(FM)$ is an integrable Lie subgroupoid of $\tilde{J}^1(FM)$. Let be $x_0, y_0 \in M$ and $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$ two local charts through x_0 and y_0 respectively such that there exist Ψ_U and Ψ_V satisfying Eq. (3.34) over the reduced Lie subgroupoid $\tilde{J}^1_{\overline{G}}(FM)$. We will also suppose that $U \cap V \neq \emptyset$, $\overline{U} = \overline{V} = B_\epsilon(0)$ and $\psi_U(x_0) = \psi_V(y_0) = 0$. Then, fixing $z \in U \cap V$ and shrinking V if it were necessary (keeping $y \in V$), we may define a new chart over V as follows:

$$\overline{\psi}_V = \tau_{-\psi_U(z)} \circ \psi_U \circ \psi_V^{-1} \circ \tau_{\psi_V(z)} \circ \psi_V.$$

Notice that, using Eq. (3.34), for all x, y we have that

$$j^1_{x,y}\left(F\left(\psi_V^{-1} \circ \tau_{\psi_V(z)} \circ \psi_V\right)\right) \tag{3.36}$$

is in $\tilde{J}^1_{\overline{G}}(FM)$. Now, let be $x \in U$, $y \in V$ and $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{G}}(FM)$. Then, using Eq. (3.36), we can prove that

$$j^1_{0,0}\left(F\left(\tau_{-\overline{\psi}_V(y)} \circ \overline{\psi}_V\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right)$$

is in $\tilde{J}^1_{\overline{G}}(FM)$. So, the charts $\overline{\psi}_V$ and ψ_U induce a diffeomorphism in the way of Eq. (3.34).

To end the proof, if $U \cap V = \emptyset$ we can find a finite family of local neighbourhoods $\{V_i\}_{i=1,\ldots,k}$ such that

(i) $U = V_1$

(ii) $V = V_k$

(iii) $V_i \cap V_{i+1} \neq \emptyset, \ \forall i$

Thus, we can find $\Psi_{U,V}$ as in the previous case. Notice that $\overline{\psi}_V$ induces a Lie groupoid isomorphism over $\tilde{J}^1_{\overline{G}}(FV)$.

Conversely, suppose that for all $x, y \in M$ there exist two open sets $U, V \subseteq M$ with $x \in U$ and $y \in V$ and two local charts $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$ which induce a diffeomorphism $\Psi_{U,V} : \tilde{J}^1_{\overline{G}}(FU, FV) \to \overline{U} \times \overline{V} \times \overline{G}$ such that for all $j^1_{x,y}\Psi$

$$\Psi_{U,V}\left(j^1_{x,y}\Psi\right) = \left(\psi_U\left(x\right), \psi_V\left(y\right), \overline{\Psi}_{U,V}\left(j^1_{x,y}\Psi\right)\right) \tag{3.37}$$

where, $\overline{\Psi}_{U,V}\left(j^1_{x,y}\Psi\right)$ is given by

$$j^1_{0,0}\left(F\left(\tau_{-\psi_V(y)} \circ \psi_V\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right).$$

Let $x_0 \in M$ be a point in M and $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$ be local charts through x_0 which induced diffeomorphism

$$\Psi_{U,V} : \tilde{J}^1_{\overline{G}}(FU, FV) \to \overline{U} \times \overline{V} \times \overline{G}.$$

For each $y \in U \cap V$, we have

$$\overline{\Psi}_{U,V}\left(j^1_{y,y}Id\right) = j^1_{0,0}\left(F\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \psi_U^{-1} \circ \tau_{\psi_U(y)}\right)\right)$$

is in $\overline{G}$. Then, for all $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{G}}(F(U \cap V))$, we deduce

$$j^1_{0,0}\left(F\left(\tau_{-\psi_U(y)} \circ \psi_U\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right) \in \overline{G}.$$

Therefore, denoting $U \cap V$ by W, the map

$$\Psi_W : \tilde{J}^1_{\overline{G}}(FW) \to \overline{W} \times \overline{W} \times \overline{G},$$

such that $\Psi_W = \left(\psi_W \circ \alpha, \psi_W \circ \beta, \overline{\Psi}_W\right)$, where ψ_W is the restriction of ψ_U to W and for each $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{G}}(FW)$ the image $\overline{\Psi}_W\left(j^1_{x,y}\Psi\right)$ is given by

$$j^1_{0,0}\left(F\left(\tau_{-\psi_W(y)} \circ \psi_W\right) \circ \Psi \circ F\left(\psi_W^{-1} \circ \tau_{\psi_W(x)}\right)\right),$$

is, indeed, a Lie groupoid isomorphism over ψ_W, and, hence, $\tilde{J}^1_{\overline{G}}(FM)$ is an integrable Lie subgroupoid of $\tilde{J}^1(FM)$.

□

Let $\overline{\mathcal{P}} : M \times M \to \tilde{J}^1(FM)$ be a section of $\left(\overline{\alpha}, \overline{\beta}\right)$. Using this result we can claim that $\overline{\mathcal{P}}$ is integrable if, and only if, for each $x, y \in M$

$$\overline{\mathcal{P}}(x,y) = j^1_{x,y}\left(F\left(\psi_V^{-1} \circ \tau_{\psi_V(y)-\psi_U(x)} \circ \psi_U\right)\right), \tag{3.38}$$

for some two local charts $(\psi_U, U), (\psi_V, V)$ on M through x and y respectively.
Equivalently, using the local coordinates given in Eq. (2.29), $\overline{\mathcal{P}}$ can be locally written as follows,

$$\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, \delta^j_i\right), \delta^j_i, 0\right), \tag{3.39}$$

Let $\tilde{J}^1_{\overline{G}}(FM)$ be a reduced subgroupoid of $\tilde{J}^1(FM)$ and $\overline{Z}^2_0 \in \overline{F}^2 M$ be a second-order non-holonomic frame at $z_0 \in M$. Then, we define

$$\overline{G} := \{\overline{Z}^{2\,-1}_0 \cdot \overline{g} \cdot \overline{Z}^2_0 \ / \ \overline{g} \in \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0}\} = \overline{Z}^{2\,-1}_0 \cdot \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0} \cdot \overline{Z}^2_0, \tag{3.40}$$

where $\left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0}$ is the isotropy group of $\tilde{J}^1_{\overline{G}}(FM)$ at z_0. Therefore, $\overline{G}$ is a Lie subgroup of $\overline{G}^2(n)$. This Lie group will be called the *associated Lie group* to $\tilde{J}^1_{\overline{G}}(FM)$.
Note that, contrarily to the case of non-holonomic $\overline{G}-$structures of second order, we do not have a unique Lie group $\overline{G}$. In fact, let $\tilde{Z}^2_0$ be a

non-holonomic frame of second order at $\tilde{z}_0$ and $\tilde{G}$ be the associated Lie group; then, if we take $\overline{L}_{z_0,\tilde{z}_0} \in \left[\tilde{J}^1_{\overline{G}}\right]^{\tilde{z}_0}_{z_0}$ we have

$$\overline{G} = \left[\left(\tilde{Z}^2_0\right)^{-1} \cdot \overline{L}_{z_0,\tilde{z}_0} \cdot \overline{Z}^2_0\right]^{-1} \cdot \tilde{G} \cdot \left[\left(\tilde{Z}^2_0\right)^{-1} \cdot \overline{L}_{z_0,\tilde{z}_0} \cdot \overline{Z}^2_0\right],$$

i.e., $\overline{G}$ and $\tilde{G}$ are conjugated subgroups of $\overline{G}^2(n)$. Notice that, by construction, if $\tilde{J}^1_{\overline{G}}(FM)$ is integrable by $\overline{G}$ (i.e. locally isomorphic to the Lie trivial Lie groupoid of $\overline{G}$ over $\mathbb{R}^n$), $\overline{G}$ can be constructed using Eq. (3.40).

Proposition 3.2.11. *A reduced subgroupoid $\tilde{J}^1_{\overline{G}}(FM)$ of $\tilde{J}^1(FM)$ is integrable if, and only if, for each point $x \in M$ there exists a (local) coordinate system $\left(x^i\right)$ on an open set $U \subseteq M$ with $x \in U$ such that the local section,*

$$\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, \delta^j_i\right), \delta^j_i, 0\right), \tag{3.41}$$

takes values into $\tilde{J}^1_{\overline{G}}(FM)$.

Proof. The proof is similar to the proof of proposition A.0.14. First, it is obvious that if $\tilde{J}^1_{\overline{G}}(FM)$ is integrable then, we can restrict the maps Ψ^{-1}_U to $\overline{U} \times \overline{U} \times \{\overline{e}\}$ to get (local) integrable sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ which takes values on $\tilde{J}^1_{\overline{G}}(FM)$.
Conversely, for each $x \in M$ there exists an open set $U \subseteq M$ with $x \in U$ and $\overline{\mathcal{P}} : U \times U \to \tilde{J}^1_{\overline{G}}(FU)$ an integrable sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ given by

$$\overline{\mathcal{P}}(x, y) = j^1_{x,y}\left(F\left(\psi^{-1}_U \circ \tau_{\psi_U(y)-\psi_U(x)} \circ \psi_U\right)\right),$$

where $\psi_U : U \to \overline{U}$ is a local chart at x.
Then, we can construct the map

$$\Psi^{-1}_U : \overline{U} \times \overline{U} \times \{\overline{e}\} \to \tilde{J}^1_{\overline{G}}(FU),$$

defined as follows

$$\Psi_U^{-1}(x,y,\overline{e}) = \overline{\mathcal{P}}\left(\psi_U^{-1}(x), \psi_U^{-1}(y)\right), \ \forall x,y \in \overline{U}.$$

Now, Let $z_0 \in U$ be a point at U, $\overline{Z}_0^2 \triangleq j_{0,z_0}^1\left(F\left(\psi_U^{-1} \circ \tau_{\psi_U(z_0)}\right)\right) \in \overline{F}^2 U$ be a non-holonomic second-order frame at z_0 and $\overline{G}$ be the Lie subgroup of $\overline{G}^2(n)$ satisfying Eq. (3.40). Then, we can define

$$\Psi_U : \tilde{J}^1_{\overline{G}}(FU) \to \overline{U} \times \overline{U} \times \overline{G},$$

where for each $j^1_{z_0,z_0}\Psi \in \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0}$ and $x,y \in \overline{U}$ we define

$$\Psi_U^{-1}\left(x, y, \overline{Z}_{00}^{2^{-1}} \cdot j^1_{z_0,z_0}\Psi \cdot \overline{Z}_0^2\right) =$$

$$= j^1_{0,\psi_U^{-1}(y)}\left(F\left(\psi_U^{-1} \circ \tau_y\right)\right) \cdot [\overline{Z}_0^{2^{-1}} \cdot j^1_{z_0,z_0}\Psi \cdot \overline{Z}_0^2] \cdot$$
$$\cdot j^1_{\psi_U^{-1}(x),0}\left(F\left(\tau_{-x} \circ \psi_U\right)\right)$$

Hence the map $\Psi_U : \overline{U} \times \overline{U} \times \overline{G} \to \tilde{J}^1_{\overline{G}}(FU)$ is an isomorphism of Lie groupoids induced by ψ_U. To prove that it is well defined we only have to take into account that

$$\begin{aligned}
& j^1_{0,\psi_U^{-1}(y)}\left(F\left(\psi_U^{-1} \circ \tau_y\right)\right) \cdot \overline{Z}_0^{2^{-1}} = \\
= \ & j^1_{z_0,\psi_U^{-1}(y)}\left(F\left(\psi_U^{-1} \circ \tau_{y-\psi_U(z_0)} \circ \psi_U\right)\right) \\
= \ & \overline{\mathcal{P}}\left(z_0, \psi_U^{-1}(y)\right) \in \tilde{J}^1_{\overline{G}}(FU), \ \forall y \in \overline{U}.
\end{aligned}$$

□

Next, we want to define the notion of *second-order non-holonomic prolongation* in $\tilde{J}^1(FM)$. In order to do this, we will define the projections $\overline{\Pi}_1^2$ and $\tilde{\Pi}_1^2$ which will be closely related with the maps $\tilde{\rho}_1^2$ and $\overline{\rho}_1^2$ (see appendix A) by the Equalities given in remark 3.2.28. Thus, we define

$$\begin{array}{rccl} \overline{\Pi}_1^2 : & \tilde{J}^1(FM) & \to & \Pi^1(M,M) \\ & j^1_{x,y}\Psi & \mapsto & \Psi(X)\left[X^{-1}\right] \end{array}$$

where $X \in FM$ is a frame at x. It is easy to show that $\overline{\Pi}_1^2$ is well-defined and, locally,

$$\overline{\Pi}_1^2\left(\left(x^i\right), \left(y^j, y^j_i\right), y^j_{,i}, y^j_{i,k}\right) = \left(x^i, y^j, y^j_i\right).$$

On the other hand we consider

$$\begin{array}{rccl} \tilde{\Pi}_1^2 : & \tilde{J}^1(FM) & \to & \Pi^1(M,M) \\ & j^1_{x,y}\Psi & \mapsto & j^1_{x,y}\psi \end{array}$$

where ψ is the induced map of Ψ over M. Then, locally

$$\tilde{\Pi}_1^2\left(\left(x^i\right), \left(y^j, y^j_i\right), y^j_{,i}, y^j_{i,k}\right) = \left(x^i, y^j, y^j_{,i}\right).$$

Notice that $\overline{\Pi}_1^2$ and $\tilde{\Pi}_1^2$ are, indeed, Lie groupoid morphims over the identity map on M. Then, let $\overline{\mathcal{P}} : M \times M \to \tilde{J}^1(FM)$ be a section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$ the projections $\mathcal{P} = \overline{\Pi}_1^2 \circ \overline{\mathcal{P}}$ and $\mathcal{Q} = \tilde{\Pi}_1^2 \circ \overline{\mathcal{P}}$ are sections of (α, β) in $\Pi^1(M,M)$.

Next, we will invert this process and, to do this, we will get inspired from remark A.0.17. Let $\mathcal{P}, \mathcal{Q} : M \times M \to \Pi^1(M,M)$ be two sections of (α, β) in $\Pi^1(M,M)$ such that

$$\mathcal{Q}(x,y) = j^1_{x,y}\psi_{xy}, \ \forall x, y \in M.$$

Thus, we constuct the following map

$$\begin{array}{rccl} \overline{\mathcal{P} \circ \psi_{xy}} : & FU & \to & FV \\ & j^1_{0,f(0)}f & \mapsto & \mathcal{P}\left(f(0), \psi_{xy}(f(0))\right) \cdot j^1_{0,f(0)}f \end{array}$$

where $\psi_{xy} : U \to V$. Analogously to remark A.0.17, $\overline{\mathcal{P} \circ \psi_{xy}}$ is a local principal bundle isomorphism with $\psi_{xy} : U \to V$ as its induced map over M. In fact, the inverse is given by

$$j^1_{0,g(0)} g \mapsto \left[\mathcal{P}\left(\psi^{-1}_{xy}\left(g\left(0\right)\right), g\left(0\right)\right)\right]^{-1} \cdot j^1_{0,g(0)} g.$$

Furthermore, let $\left(x^i\right)$ be a local coordinate system on an open set $U \subseteq M$ and $\left(x^i, x^i_j\right)$ its induced coordinates over FU, then we have

$$\overline{\mathcal{P} \circ \psi_{xy}}\left(x^i, x^i_j\right) = \left(\psi_{xy}\left(x^i\right), P^j_l\left(x^i, \psi_{xy}\left(x^i\right)\right) x^l_i\right),$$

where for each another local coordinate system $\left(y^j\right)$ on an open set $V \subseteq M$

$$\mathcal{P}\left(x^i, y^j\right) = \left(x^i, y^j, P^j_i\left(x^i, y^j\right)\right).$$

Thus, we define

$$\begin{array}{rccl} \mathcal{P}^1\left(\mathcal{Q}\right): & M \times M & \to & \tilde{J}^1\left(FM\right) \\ & (x,y) & \mapsto & j^1_{x,y}\left(\overline{\mathcal{P} \circ \psi_{xy}}\right) \end{array} \tag{3.42}$$

where we are considering the equivalence class in $\tilde{J}^1\left(FM\right)$. Notice that $\mathcal{P}^1\left(\mathcal{Q}\right)$ does not depend on ψ_{xy} because of $\mathcal{Q}$ does not depend on ψ_{xy}. $\mathcal{P}^1\left(\mathcal{Q}\right)$ will be called *second-order non-holonomic prolongation of* $\mathcal{P}$ *and* $\mathcal{Q}$ and satisfies that

(i) For all $x, y \in M$ and $j^1_{0,x} f \in FM$,

$$\begin{array}{rl} & \overline{\Pi}^2_1 \circ \mathcal{P}^1\left(\mathcal{Q}\right)(x,y) = \\ = & \left[\mathcal{P}\left(x, \psi_{xy}\left(x\right)\right) \cdot j^1_{0,x} f\right] \cdot \left(j^1_{0,x} f\right)^{-1} \\ = & \mathcal{P}\left(x,y\right). \end{array}$$

(ii) For all $x, y \in M$,

$$\tilde{\Pi}^2_1 \circ \mathcal{P}^1\left(\mathcal{Q}\right)(x,y) = j^1_{x,y}\psi_{xy} = \mathcal{Q}\left(x,y\right).$$

In fact, let be $\left(x^i\right)$ and $\left(y^j\right)$ local coordinate systems on open sets $U, V \subseteq M$ and $\left(x^i, x^i_j\right)$ and $\left(y^j, y^j_i\right)$ its induced coordinates over FU and FV respectively, then we have

$$\mathcal{P}^1\left(\mathcal{Q}\right)\left(x^i, y^j\right) = \left(\left(x^i, y^j, P^j_i\right), Q^j_i, R^j_{i,k}\right),$$

where

$$\mathcal{P}\left(x^i, y^j\right) = \left(x^i, y^j, P^j_i\right), \qquad \mathcal{Q}\left(x^i, y^j\right) = \left(x^i, y^j, Q^j_i\right).$$

Furthermore, for each $k = 1, \cdots, n$, we obtain

$$\begin{aligned} & \frac{\partial\left(P^j_i \circ \left(Id_U, \psi_{xy}\right)\right)}{\partial x^k_{|x}} = \\ = \;& dP^j_{i\ |(x,y)} \circ \frac{\partial\left(Id_U, \psi_{xy}\right)}{\partial x^k_{|x}} \\ = \;& dP^j_{i\ |(x,y)}\left(x^k, \left(Q^1_k\left(x,y\right), \cdots, Q^n_k\left(x,y\right)\right)\right) \\ = \;& d\left(P^j_i\right)_{y\,|x}\left(x^k\right) + d\left(P^j_i\right)_{x\,|y}\left(Q^1_k\left(x,y\right), \cdots, Q^n_k\left(x,y\right)\right) \\ = \;& \frac{\partial\left(P^j_i\right)_y}{\partial x^k_{|x}} + d\left(P^j_i\right)_{x\,|y}\left(Q^1_k\left(x,y\right), \cdots, Q^n_k\left(x,y\right)\right), \end{aligned}$$

where we are fixing the first (resp. the second) coordinate when we write $\left(P^j_i\right)_x$ (resp. $\left(P^j_i\right)_y$). Then, by definition of induced coordinates, $R^j_{i,k}$ is given by

$$R^j_{i,k}\left(x,y\right) = \frac{\partial\left(P^j_i\right)_y}{\partial x^k_{|x}} + Q^l_k\left(x,y\right)\frac{\partial\left(P^j_i\right)_x}{\partial y^l_{|y}}.$$

We will denote this expression by

$$R^j_{i,k} = \frac{\partial P^j_i}{\partial x^k} + Q^l_k \frac{\partial P^j_i}{\partial y^l}.$$

Then, any section $\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, P^j_i\right), Q^j_i, R^j_{i,k}\right)$ of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$ which projects onto $\mathcal{P}\left(x^i, y^j\right) = \left(x^i, y^j, P^j_i\right)$ and $\mathcal{Q}\left(x^i, y^j\right) = \left(x^i, y^j, Q^j_i\right)$ via $\overline{\Pi}^2_1$ and $\tilde{\Pi}^2_1$ respectively is a prolongation if, and only if,

$$R^j_{i,k} = \frac{\partial P^j_i}{\partial x^k} + Q^l_k \frac{\partial P^j_i}{\partial y^l}. \tag{3.43}$$

Thus, we have established the notion of prolongation in the second-order non-holonomic groupoid, and we can give the following definition.

Definition 3.2.12. Let $\mathcal{P}^1(\mathcal{Q})$ be a non-holonomic prolongation of second order in $\tilde{J}^1(FM)$. $\mathcal{P}^1(\mathcal{Q})$ is said to be *integrable* in $\tilde{J}^1(FM)$ if $\mathcal{Q}$ is an integrable section of $\Pi^1(M, M)$.

Notice that, using the introduced coordinates, an integrable prolongation can be seen locally as follows

$$\mathcal{P}^1(\mathcal{Q})\left(x^i, y^j\right) \quad = \quad \left(\left(x^i, y^j, P^j_i\right), \delta^j_i, \frac{\partial P^j_i}{\partial x^k} + \frac{\partial P^j_i}{\partial y^k}\right)$$

Thus, as in the case of second-order non-holonomic frame bundle, we have two remarkable sections: integrable sections and integrable prolongations. So, it is easy to give the following result (similar to proposition A.0.18) which will help us to understand why the integrable prolongation are not necessarily integrable as sections.

Proposition 3.2.13. *Let $\overline{\mathcal{P}}$ be a section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$. $\overline{\mathcal{P}}$ is integrable if, and only if, $\overline{\mathcal{P}} = \mathcal{P}^1(\mathcal{Q})$ is a second-order non-holonomic integrable prolongation and $\mathcal{P} = \mathcal{Q}$. In particular, a second-order non-holonomic integrable prolongation $\mathcal{P}^1(\mathcal{Q})$ is integrable if, and only if, $\mathcal{P}^1(\mathcal{Q})$ takes values in $\tilde{j}^1(FM)$.*

Thus, second-order non-holonomic integrable prolongations can be seen as a natural generalization of integrable sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ (see proposition 3.2.14).

Notice that, analogously to Eq. (A.10), we can prove the following result.

Proposition 3.2.14. *Let $\overline{\mathcal{P}}$ be a section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$. $\overline{\mathcal{P}}$ is a second-order non-holonomic integrable prolongation if, and only if, for all $x_0, y_0 \in M$ there exist two open sets $U, V \subseteq M$ with $x_0 \in U$ and $y_0 \in V$ and two local principal bundle isomorphisms $\Psi : FV \to F\overline{V}$ and $\Phi : FU \to F\overline{U}$ such that*

$$\overline{\mathcal{P}}(x, y) = j^1_{x,y}\left(\Psi^{-1} \circ F\tau_{\psi(y)-\phi(x)} \circ \Phi\right), \ \forall (x, y) \in U \times V. \tag{3.44}$$

Proof. Suppose that $\overline{\mathcal{P}} = \mathcal{P}^1(\mathcal{Q})$ is a non-holonomic integrable prolongation of second order. Take $z_0 \in M$ and $\overline{Z}_0^2 \in \overline{F}^2 M$ with $\overline{\rho}^2\left(\overline{Z}_0^2\right) = z_0$ such that the section of FM given by

$$Q_{z_0}(y) = \mathcal{Q}(z_0, y) \cdot \tilde{\rho}^2_1\left(\overline{Z}_0^2\right),$$

is integrable.
Considering the section of FM which satisfies that

$$P_{z_0}(y) = \mathcal{P}(z_0, y) \cdot \overline{\rho}^2_1\left(\overline{Z}_0^2\right),$$

we can define

$$\overline{P} = P^1_{z_0}(Q_{z_0}),$$

which is a non-holonomic integrable prolongation of second order on M. Finally, we can prove that for all $x, y \in M$, we have

$$\overline{\mathcal{P}}(x, y) = \overline{P}(y)\,[\overline{P}(x)^{-1}].$$

This will be explained with more detail in the last section (see Eq. (3.59)). Now, let be $x, y \in M$. We use Eq. (A.10) to ensure that there exist two open sets $U, V \subseteq M$ with $x_0 \in U$ and $y_0 \in V$ and two local principal bundle isomorphisms $\Psi : FV \to F\overline{V}$ and $\Phi : FU \to F\overline{U}$ such that for all $(x, y) \in U \times V$,

$$P^1_{z_0}(Q_{z_0})(y) = j^1_{0,y}\left(\Phi^{-1} \circ F\tau_{\phi(y)}\right),$$

$$P^1_{z_0}(Q_{z_0})(x) = j^1_{0,x}\left(\Psi^{-1} \circ F\tau_{\psi(x)}\right).$$

Therefore,

$$\overline{\mathcal{P}}(x, y) = j^1_{x,y}\left(\Psi^{-1} \circ F\tau_{\psi(y)-\phi(x)} \circ \Phi\right),$$

for all $(x, y) \in U \times V$.
Conversely, take $x_0, y_0 \in M$; then, there exist two open sets $U, V \subseteq M$ with $x_0 \in U$ and $y_0 \in V$ and two local principal bundle isomorphisms $\Psi : FV \to F\overline{V}$ and $\Phi : FU \to F\overline{U}$ such that

$$\overline{\mathcal{P}}(x, y) = j^1_{x,y}\left(\Psi^{-1} \circ F\tau_{\psi(y)-\phi(x)} \circ \Phi\right), \ \forall (x, y) \in U \times V.$$

Now, we only have to take as local charts the induced maps of Ψ and Φ over M to prove the result. In fact,

$$\Psi^{-1} \circ F\tau_{\psi(y)-\phi(x)} \circ \Phi = \overline{\mathcal{P} \circ \left(\psi^{-1} \circ \tau_{\psi(y)-\phi(x)} \circ \phi\right)},$$

for all $(x, y) \in U \times V$, where

$$\mathcal{P} = \overline{\Pi}^2_1 \circ \overline{\mathcal{P}}.$$

□

Now, we will extend this concept to reduced subgroupoids.

Definition 3.2.15. Let $\tilde{J}^1_{G}(FM)$ be a reduced subgroupoid of $\tilde{J}^1(FM)$. $\tilde{J}^1_{G}(FM)$ is an *integrable prolongation* if can be covered M with local integrable prolongations which take values in $\tilde{J}^1_{G}(FM)$.

Proposition 3.2.16. *Let $\tilde{J}^1_{G}(FM)$ be an integrable prolongation. $\tilde{J}^1_{G}(FM)$ is integrable if, and only if, $\tilde{J}^1_{G}(FM)$ is contained in $\tilde{j}^1(FM)$.*

Notice that, definition 3.2.15 can be expressed as follows: For any point $x \in M$ there exists a local coordinate system $\left(x^i\right)$ over an open set $U \subseteq M$ which contains x such that there is a local section

$$\mathcal{P}^1 \left(\mathcal{Q}\right)\left(x^i, y^j\right) = \left(\left(x^i, y^j, P_i^j\right), \delta_i^j, \frac{\partial P_i^j}{\partial x^k} + \frac{\partial P_i^j}{\partial y^k}\right), \tag{3.45}$$

which takes values in $\tilde{J}^1_{\overline{G}}\left(FM\right)$.

Remark 3.2.17. Analogously to remark A.0.20, let $\tilde{J}^1_{\overline{G}}\left(FM\right)$ be a reduced subgroupoid of $\tilde{J}^1\left(FM\right)$. We can prove that $\tilde{J}^1_{\overline{G}}\left(FM\right)$ is an integrable prolongation if, and only if, for each point $x \in M$, there exists a local isomorphism of principal bundles, $\Psi_U : FU \rightarrow F\overline{U}$, with $x \in U$ such that induces an isomorphism of Lie groupoids given by

$$\Upsilon_U : \tilde{J}^1_{\overline{G}}\left(FU\right) \rightarrow \overline{U} \times \overline{U} \times \overline{G},$$

where $\Upsilon_U\left(j^1_{x,y}H\right) = \left(\psi_U\left(x\right), \psi_U\left(y\right), \overline{\Upsilon}_U\left(j^1_{x,y}H\right)\right)$ and

$$\overline{\Upsilon}_U\left(j^1_{x,y}H\right) = j^1_{0,0}\left(F\left(\tau_{-\psi_U(y)}\right) \circ \Psi_U \circ H \circ \Psi_U^{-1} \circ F\left(\tau_{\psi_U(x)}\right)\right),$$

with ψ_U is the induced map of Ψ_U over the base manifold.

So, in a similar way to proposition 3.2.10, we may prove the following:

Proposition 3.2.18. *Let $\tilde{J}^1_{\overline{G}}\left(FM\right)$ be a reduced Lie subgroupoid of $\tilde{J}^1\left(FM\right)$. $\tilde{J}^1_{\overline{G}}\left(FM\right)$ is an integrable prolongation if, and only if, for all $x, y \in M$ there exist two open sets $U, V \subseteq M$ with $x \in U$ and $y \in V$ and two local isomorphisms $\Psi_U : FU \rightarrow F\overline{U}$ and $\Psi_V : FV \rightarrow F\overline{V}$ which induce the following isomorphism of Lie groupoids:*

$$\Upsilon_{U,V} : \tilde{J}^1_{\overline{G}}\left(FU, FV\right) \rightarrow \overline{U} \times \overline{V} \times \overline{G}, \tag{3.46}$$

where $\Upsilon_{U,V}\left(j^1_{x,y}H\right) = \left(\psi_U\left(x\right), \psi_V\left(y\right), \overline{\Upsilon}_{U,V}\left(j^1_{x,y}H\right)\right)$ and

$$\overline{\Upsilon}_{U,V}\left(j^1_{x,y}H\right) = j^1_{0,0}\left(F\left(\tau_{-\psi_V(y)}\right) \circ \Psi_V \circ H \circ \Psi_U^{-1} \circ F\left(\tau_{\psi_U(x)}\right)\right),$$

with ψ_U and ψ_V are the induced map of Ψ_U and Ψ_V over the base manifold respectively.

Hence, we have that: $\tilde{J}^1_{\overline{G}}(FM)$ is an integrable prolongation if, and only if, for any two point $x, y \in M$ there exist two local coordinate systems $\left(x^i\right)$ and $\left(y^j\right)$ over x and y respectively such that there is a local section

$$\mathcal{P}^1(\mathcal{Q})\left(x^i, y^j\right) = \left(\left(x^i, y^j, P^j_i\right), \delta^j_i, \frac{\partial P^j_i}{\partial x^k} + \frac{\partial P^j_i}{\partial y^k}\right), \tag{3.47}$$

which takes values in $\tilde{J}^1_{\overline{G}}(FM)$.

Now, we will translate these results to the associated Lie algebroid. Thus, we will express the notions of integrability over the second-order non-holonomic algebroid over an manifold M. We will begin defining the notion of *integrability of a reduced Lie subalgebroid*. In order to do that, we will denote by $\overline{\mathfrak{g}}^2(n)$ the associated Lie algebra of $\overline{G}^2(n)$.

Definition 3.2.19. Let $A\tilde{J}^1_{\overline{G}}(FM)$ be a reduced Lie subalgebroid of $A\tilde{J}^1(FM)$. $A\tilde{J}^1_{\overline{G}}(FM)$ is *integrable by* $\overline{G}$ if it is locally isomorphic to the trivial algebroid $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \overline{\mathfrak{g}})$ where $\overline{\mathfrak{g}}$ is the Lie subalgebra of $\overline{\mathfrak{g}}^2(n)$.

We will consider $\overline{G}$ as the unique Lie subgroup of $\overline{G}^2(n)$ whose associated Lie algebra is $\overline{\mathfrak{g}}$.
Note that $A\tilde{J}^1_{\overline{G}}(FM)$ is locally isomorphic to $T\mathbb{R}^n \oplus (\mathbb{R}^n \times \overline{\mathfrak{g}})$ if for all $x \in M$ there exists an open set $U \subseteq M$ with $x \in U$ and a local chart, $\psi_U : U \to \overline{U}$, which induces an isomorphism of Lie algebroids,

$$A\Psi_U : A\tilde{J}^1_{\overline{G}}(FU) \to T\overline{U} \oplus \left(\overline{U} \times \overline{\mathfrak{g}}\right), \tag{3.48}$$

where $A\Psi_U$ is the induced map of the isomorphism of Lie groupoids Ψ_U which is given by

$$\Psi_U : \tilde{J}^1_{\overline{G}}(FU) \to \overline{U} \times \overline{U} \times \overline{G},$$

such that $\Psi_U = \left(\psi_U \circ \overline{\alpha}, \psi_U \circ \overline{\beta}, \overline{\Psi}_U\right)$, where for each $j^1_{x,y}\Psi \in \tilde{J}^1_{\overline{G}}(FU)$ the image $\overline{\Psi}_U\left(j^1_{x,y}\Psi\right)$ is given by

$$j^1_{0,0}\left(F\left(\tau_{-\psi_U(y)} \circ \psi_U\right) \circ \Psi \circ F\left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)\right), \tag{3.49}$$

for some Lie subgroupoid $\tilde{J}^1_{\overline{G}}(FU)$ of $\tilde{J}^1(FU)$.

So, for each open subset $U \subseteq M$, $A\tilde{J}^1_{\overline{G}}(FU)$ is integrable by a Lie subgroupoid $\tilde{J}^1_{\overline{G}}(FU)$ of $\tilde{J}^1(FU)$. Using the uniqueness of integrating immersed (source-connected) subgroupoids (see proposition 2.3.41), $A\tilde{J}^1_{\overline{G}}(FM)$ is integrable by a Lie subgroupoid of $\tilde{J}^1(FM)$ which will be denoted by $\tilde{J}^1_{\overline{G}}(FM)$. Obviously, $A\tilde{J}^1_{\overline{G}}(FM)$ is integrable if, and only if, $\tilde{J}^1_{\overline{G}}(FM)$ is integrable.

Analogously to the case of the 1−jets groupoid, a *parallelism* of $A\tilde{J}^1(FM)$ is an associated Lie algebroid of a parallelism of $\tilde{J}^1(FM)$. Hence, using the Lie's second fundamental theorem 2.3.40, a parallelism is a section of $\overline{\sharp}$, where $\overline{\sharp}$ is the anchor of $A\tilde{J}^1(FM)$ (understanding "section" as section in the category of Lie algebroids, i.e., Lie algebroid morphism from the tangent algebroid TM to $A\tilde{J}^1(FM)$ which is a section of the morphism $\overline{\sharp}$), and conversely. In this way, we will also speak about *integrable sections of* $\overline{\sharp}$.

Let $\left(x^i\right)$ be a local coordinate system defined on some open subset $U \subseteq M$, then, we will use the local coordinate system defined in Eq. (2.50),

$$A\tilde{J}^1(FU) : \left(x^i, v^i, v^i_j, v^i_{,j}, v^i_{j,k}\right) \tag{3.50}$$

which are, indeed, induced coordinates by the functor from Lie groupoid to Lie algebroids (see theorem 2.3.26) of local coordinates on $\tilde{J}^1(FU)$.

Notice that each integrable section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$, $\overline{\mathcal{P}}$, is a Lie groupoid morphism. Hence, $\overline{\mathcal{P}}$ induces a Lie algebroid morphism $A\overline{\mathcal{P}} : TM \to A\tilde{J}^1(FM)$ which is a section of $\overline{\sharp}$ and is given (see Eq. (2.46)) by

$$A\overline{\mathcal{P}}(v_x) = T_x\overline{\mathcal{P}}_x(v_x), \ \forall v_x \in T_xM, \tag{3.51}$$

where $\overline{\mathcal{P}}_x : M \to \tilde{J}^1 x(FM)$ satisfies that

$$\overline{\mathcal{P}}_x(y) = \overline{\mathcal{P}}(y, x), \forall x, y \in M.$$

So, taking into account Eq. (3.39), locally,

$$\overline{\mathcal{P}}\left(x^{i},y^{j}\right)=\left(\left(x^{i},y^{j},\delta_{i}^{j}\right),\delta_{i}^{j},0\right),$$

and we have that each integrable section can be written locally as follows

$$A\overline{\mathcal{P}}\left(x^{i},\frac{\partial}{\partial x^{i}}\right)=\left(\left(x^{i},\frac{\partial}{\partial x^{i}},0\right),0,0\right).$$

Now, using proposition 3.2.11, we deduce the following analogous proposition.

Proposition 3.2.20. *A reduced subalgebroid $A\tilde{J}^{1}_{\overline{G}}(FM)$ of $A\tilde{J}^{1}(FM)$ is integrable by $\overline{G}$ if, and only if, there exist local integrable sections of $\overline{\sharp}$ covering M which takes values on $A\tilde{J}^{1}_{\overline{G}}(FM)$.*

Equivalently, for each point $x\in M$ there exists a local coordinate system $\left(x^{i}\right)$ over an open set $U\subseteq M$ with $x\in U$ such that the local sections

$$\Lambda\left(x^{i},\frac{\partial}{\partial x^{i}}\right)=\left(\left(x^{i},\frac{\partial}{\partial x^{i}},0\right),0,0\right),$$

takes values in $A\tilde{J}^{1}_{\overline{G}}(FM)$.

Next, we will have to introduce the notion of prolongation over the induced Lie algebroid $A\tilde{J}^{1}(FM)$. In this way, taking into account that $\overline{\Pi}^{2}_{1}$ and $\tilde{\Pi}^{2}_{1}$ are morphisms of Lie groupoids we can consider the induced morphisms of Lie algebroids $A\overline{\Pi}^{2}_{1}, A\tilde{\Pi}^{2}_{1}: A\tilde{J}^{1}(FM)\to A\Pi^{1}(M,M)$. Thus, it is easy to induce the construction of the second-order non-holonomic prolongation over $A\tilde{J}^{1}(FM)$. Given two section of $\sharp$

$$A\mathcal{P},A\mathcal{Q}:TM\to A\Pi^{1}(M,M),$$

we define the *second-order non-holonomic prolongation associated to $A\mathcal{P}$ and $A\mathcal{Q}$* as follows,

$$A\mathcal{P}^{1}(A\mathcal{Q})=A\left(\mathcal{P}^{1}(\mathcal{Q})\right).$$

Then, $A\mathcal{P}^1(A\mathcal{Q})$ projects via $A\overline{\Pi}_1^2$ (resp. $A\tilde{\Pi}_1^2$) over $A\mathcal{P}$ (resp. $A\mathcal{Q}$).
Using that the functor A preserves integrability (see section 3.1), $A\mathcal{P}^1(A\mathcal{Q})$ is said to be *integrable* if $A\mathcal{Q}$ is integrable (equivalently $\mathcal{Q}$ is integrable). Therefore, if $A\mathcal{P}^1(A\mathcal{Q})$ takes values in $A\tilde{j}^1(FM)$, $A\mathcal{P}^1(A\mathcal{Q})$ is an integrable prolongation if, and only if, it is integrable.
Finally, we can introduce the following definition.

Definition 3.2.21. Let $A\tilde{J}^1_{\overline{G}}(FM)$ be a Lie subalgebroid of $A\tilde{J}^1(FM)$. $A\tilde{J}^1_{\overline{G}}(FM)$ is a non-holonomic integrable prolongation of second-order if we can cover M by local non-holonomic integrable prolongations of second order which take values in $A\tilde{J}^1_{\overline{G}}(FM)$.

Remark 3.2.22. Thus, $A\tilde{J}^1_{\overline{G}}(FM)$ is a non-holonomic integrable prolongation of second-order if, and only if, $\tilde{J}^1_{\overline{G}}(FM)$ is a non-holonomic integrable prolongation of second-order. Notice that, if $\tilde{J}^1_{\overline{G}}(FM)$ is a non-holonomic integrable prolongation of second-order then, we can cover M by open sets U and second-order non-holonomic integrable prolongations $\mathcal{P}^1(\mathcal{Q}) : U \times U \to \tilde{J}^1_{\overline{G}}(FU)$. However, we cannot take $A\mathcal{P}^1(\mathcal{Q})$ because these sections are not morphisms of Lie groupoids.
To solve this, we fix $z_0 \in M$ and define

$$\mathcal{P}^1(\mathcal{Q})^{z_0}(x,y) = \mathcal{P}^1(\mathcal{Q})(z_0,y) \cdot \left[\mathcal{P}^1(\mathcal{Q})(z_0,x)\right]^{-1},$$

for all $x, y \in U$. Then, these family of sections are morphisms of Lie groupoids and non-holonomic integrable prolongations of second-order.

Now, express this condition in local coordinates. Let $\overline{\mathcal{P}} : M \times M \to \tilde{J}^1(FM)$ be a section of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\tilde{J}^1(FM)$ and $\left(x^i\right)$ be a local coordinate system on M such that

$$\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, P_i^j\right), Q_i^j, R_{i,k}^j\right).$$

Then,

$$A\overline{\mathcal{P}}\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(\left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right), \frac{\partial Q_i^j}{\partial x^l}, \frac{\partial R_{i,k}^j}{\partial x^l}\right),$$

where we are differentiating fixing the first coordinate (see Eq. (3.51)). Next, take two section of $\sharp$, $A\mathcal{P}$ and $A\mathcal{Q}$, in $A\Pi^1(M,M)$ such that

$$A\mathcal{P}\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right)$$

$$A\mathcal{Q}\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial Q_i^j}{\partial x^l}\right).$$

Hence,

$$A\mathcal{P}^1(A\mathcal{Q})\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(\left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right), \frac{\partial Q_i^j}{\partial x^l}, R_{i,kl}^j\right), \tag{3.52}$$

where

$$R_{i,kl}^j = \frac{\partial^2 P_i^j}{\partial x^l \partial x^k} + \frac{\partial Q_k^m}{\partial x^l}\frac{\partial P_i^j}{\partial y^m} + Q_k^m \frac{\partial^2 P_i^j}{\partial x^l \partial y^m}.$$

To understand why we obtain this local expression we have to take into account that we are fixing the second coordinate to get the induced map $A\mathcal{P}^1(A\mathcal{Q})$. Finally, using Eq. (3.52), $A\mathcal{P}^1(A\mathcal{Q})$ is integrable if, and only if, there exists a local system of coordinates (x^i) such that $A\mathcal{P}^1(A\mathcal{Q})\left(x^i, \frac{\partial}{\partial x^l}\right)$ can be expressed as follows

$$\left(\left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right), 0, \frac{\partial^2 P_i^j}{\partial x^l \partial x^k} + \frac{\partial^2 P_i^j}{\partial x^l \partial y^k}\right). \tag{3.53}$$

So, we can rewrite definition 3.2.21 in the following way: Let $A\tilde{J}^1_{\overline{G}}(FM)$ be a Lie subalgebroid of $A\tilde{J}^1(FM)$. $A\tilde{J}^1_{\overline{G}}(FM)$ is a non-holonomic integrable prolongation of second-order if for each $x \in M$ there exists a local coordinate system (x^i) over x such that the local section $A\mathcal{P}^1(A\mathcal{Q})\left(x^i, \frac{\partial}{\partial x^l}\right)$ of $\overline{\sharp}$,

$$\left(\left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right), 0, \frac{\partial^2 P_i^j}{\partial x^l \partial x^k} + \frac{\partial^2 P_i^j}{\partial x^l \partial y^k}\right),$$

take values in $A\tilde{J}^1_G(FM)$.

Characterization of homogeneity

In this subsection we will use the results obtained in the previous subsections to interpret the (local) homogeneity of Cosserat media in several different ways. So, let $F\mathcal{B}$ be a Cosserat medium with $W : \tilde{J}^1(F\mathcal{B}) \to V$ as mechanical response. Consider $\overline{\Omega}(\mathcal{B})$ the corresponding non-holonomic material groupoid of second order. Then, $\mathcal{B}$ is locally homogeneous if, and only if, for each point $X \in \mathcal{B}$ there exists an open subset $\mathcal{U} \subseteq \mathcal{B}$ with $X \in \mathcal{U}$ and a local diffeomorphism $\tilde{\kappa}$ over $\mathcal{U}$ such that the (local) section $\overline{\mathcal{P}} : \mathcal{U} \times \mathcal{U} \to \tilde{J}^1(F\mathcal{B})$ given by

$$\overline{\mathcal{P}}(Z, Y) = j^1_{Z,Y}\left(\tilde{\kappa}^{-1} \circ F\tau_{\kappa(Y)-\kappa(Z)} \circ \tilde{\kappa}\right),$$

where $\tau_{\kappa(Y)-\kappa(Z)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\kappa(Y) - \kappa(Z)$ takes values in $\overline{\Omega}(\mathcal{B})$ (see definition 3.2.5).
So, using proposition 3.2.14, we immediately have

Proposition 3.2.23. *Let $\mathcal{B}$ be a Cosserat medium. If $\mathcal{B}$ is homogeneous then $\overline{\Omega}(\mathcal{B})$ is a second-order non-holonomic integrable prolongation. In fact, $\overline{\Omega}(\mathcal{B})$ is a second-order non-holonomic integrable prolongation if, and only if, $\mathcal{B}$ is locally homogeneous.*

Therefore, we deduce that $\mathcal{B}$ is locally homogeneous if, and only if, $\Omega(\mathcal{B})$ can be locally covered by (local) sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ in $\overline{\Omega}(\mathcal{B})$ as follows:

$$\overline{\mathcal{P}}\left(x^i, y^j\right) = \left(\left(x^i, y^j, P^j_i\right), \delta^j_i, \frac{\partial P^j_i}{\partial x^k} + \frac{\partial P^j_i}{\partial y^k}\right).$$

Next, let us consider the induced subalgebroid of the second-order non-holonomic material groupoid, $A\overline{\Omega}(\mathcal{B})$. This Lie algebroid will be called *second-order non-holonomic material algebroid of* $\mathcal{B}$.

Take $\Theta \in \Gamma\left(A\overline{\Omega}\left(\mathcal{B}\right)\right)$. So, the flow of the left-invariant vector field X_{Θ}, $\{\varphi_t^{\Theta}\}$, can be restricted to $\overline{\Omega}\left(\mathcal{B}\right)$.
Hence, for any $g \in \tilde{J}^1\left(F\mathcal{B}\right)$, we have

$$W\left(\varphi_t^{\Theta}\left(g \cdot \overline{\epsilon}\left(\overline{\alpha}\left(g\right)\right)\right)\right) = W\left(g\right). \tag{3.54}$$

Thus, for each $g \in \tilde{J}^1\left(F\mathcal{B}\right)$, we deduce

$$TW\left(X_{\Theta}\left(g\right)\right) = \frac{\partial}{\partial t_{|0}}\left(W\left(\varphi_t^{\Theta}\left(g\right)\right)\right) = \frac{\partial}{\partial t_{|0}}\left(W\left(g\right)\right) = 0.$$

Therefore,

$$TW\left(X_{\Theta}\right) = 0. \tag{3.55}$$

Conversely, suppose that Eq. (3.55) is satisfied. Then,

$$\frac{\partial}{\partial t_{|s}}\left(W\left(\varphi_t^{\Theta}\left(g\right)\right)\right) = 0, \ \forall g \in \tilde{J}^1\left(F\mathcal{B}\right), \ \forall s.$$

Thus, taking into account that

$$W\left(\varphi_0^{\Theta}\left(g\right)\right) = W\left(g\right),$$

we have

$$W\left(\varphi_t^{\Theta}\left(g\right)\right) = W\left(g\right),$$

i.e.,

$$\Theta \in \Gamma\left(A\overline{\Omega}\left(\mathcal{B}\right)\right).$$

As a consequence, the second-order non-holonomic material algebroid can be defined without using the non-holonomic material groupoid of second order. Thus, we can characterize the homogeneity and uniformity using the material Lie algebroid. Taking into account that the fact of being an "integrable prolongation" can be equivalently defined over the associated Lie algebroid (see remark 3.2.22) we get the following result:

Proposition 3.2.24. *Let $\mathcal{B}$ be a Cosserat continuum. If $\mathcal{B}$ is homogeneous, then, $A\overline{\Omega}\left(\mathcal{B}\right)$ is an integrable non-holonomic prolongation of second order. Conversely, $A\overline{\Omega}\left(\mathcal{B}\right)$ is an integrable non-holonomic prolongation of second order implies that $\mathcal{B}$ is locally homogeneous.*

Using the local expression (3.53), this result can be expressed locally as follows.

Proposition 3.2.25. *Let $\mathcal{B}$ be a Cosserat continuum. $\mathcal{B}$ is locally homogeneous if, and only if, for each body point $X \in \mathcal{B}$ there exists a local coordinate system $\left(x^i\right)$ over $\mathcal{U} \subseteq \mathcal{B}$ with $X \in \mathcal{U}$ such that the local section $A\mathcal{P}^1\left(A\mathcal{Q}\right)\left(x^i, \dfrac{\partial}{\partial x^l}\right)$ of $\overline{\sharp}$,*

$$\left(\left(x^i, \frac{\partial}{\partial x^l}, \frac{\partial P_i^j}{\partial x^l}\right), 0, \frac{\partial^2 P_i^j}{\partial x^l \partial x^k} + \frac{\partial^2 P_i^j}{\partial x^l \partial y^k}\right),$$

takes values in $A\overline{\Omega}\left(\mathcal{B}\right)$.

Finally, we will use the Lie algebroid morphism $\tilde{\mathfrak{D}}$ 3.30 to give another characterization of the homogeneity. Indeed, let be $\overline{\Delta} : T\mathcal{B} \to A\tilde{J}^1\left(F\mathcal{B}\right)$ a linear section of $\overline{\sharp}$. Then, using remark 3.2.7, $\overline{\Delta}$ can be seen as a map

$$\nabla^{\overline{\Delta}} : \mathfrak{X}\left(\mathcal{B}\right) \times \mathfrak{X}\left(F\mathcal{B}\right) \to \mathfrak{X}\left(F\mathcal{B}\right),$$

where, for all $\left(\chi, \overline{\varsigma}\right) \in \mathfrak{X}\left(\mathcal{B}\right) \times \mathfrak{X}\left(F\mathcal{B}\right)$, $f \in \mathcal{C}^{\infty}\left(\mathcal{B}\right)$ and $F \in \mathcal{C}^{\infty}\left(F\mathcal{B}\right)$ satisfies that

(i) $\nabla^{\overline{\Delta}}_{f\sigma}\overline{\varsigma} = \left(f \circ \pi_{\mathcal{B}}\right)\nabla^{\overline{\Delta}}_{\chi}\overline{\varsigma}$.

(ii) $\nabla^{\overline{\Delta}}_{\chi}F\overline{\varsigma} = F\nabla^{\overline{\Delta}}_{\chi}\overline{\varsigma} + \overline{\Delta}\left(\chi\right)^{\sharp}\left(F\right)\overline{\varsigma}$.

(iii) The base vector field of $\nabla^{\overline{\Delta}}_{\chi}$ is $\overline{\Delta}\left(\chi\right)^{\sharp}$ which is $\pi_{\mathcal{B}}$−related to χ.

(iv) For all $g \in Gl\left(3, \mathbb{R}\right)$,

$$\nabla^{\overline{\Delta}}_{\chi} \circ R_g^* = R_g^* \circ \nabla^{\overline{\Delta}}_{\chi}.$$

(v) The flow of $\nabla^{\overline{\Delta}}_{\chi}$ is the tangent map of an automorphism of frame bundles (over the identity map) at each fibre.

Let (x^i) be a local coordinate system on $\mathcal{B}$ such that

$$\overline{\Delta}\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(x^i, \frac{\partial}{\partial x^l}, \Delta^j_{il}, \Delta^j_{,il}, \Delta^j_{i,kl}\right).$$

Then, $\nabla^{\overline{\Delta}}$ is locally characterized as follows

(i) $\nabla^{\overline{\Delta}}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Delta^k_{,ij} \frac{\partial}{\partial x^k} + \Delta^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla^{\overline{\Delta}}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i_j} = \Delta^l_{ik} \frac{\partial}{\partial x^l_j}$

In this way, if $\overline{\Delta} = A\mathcal{P}^1(A\mathcal{Q})$ is a non-holonomic prolongation of second order we have that

(i) $\nabla^{A\mathcal{P}^1(A\mathcal{Q})}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \frac{\partial Q^k_i}{\partial x^j} \frac{\partial}{\partial x^k} + R^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla^{A\mathcal{P}^1(A\mathcal{Q})}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i_j} = \frac{\partial P^l_i}{\partial x^k} \frac{\partial}{\partial x^l_j}$,

where

$$R^k_{l,ij} = \frac{\partial^2 P^k_l}{\partial x^j \partial x^i} + \frac{\partial Q^m_i}{\partial x^j} \frac{\partial P^k_l}{\partial x^m} + Q^m_i \frac{\partial^2 P^k_l}{\partial x^j \partial x^m}.$$

Hence, $\overline{\Delta}$ is an integrable non-holonomic prolongation of second order if an only if

(i) $\nabla^{A\mathcal{P}^1(A\mathcal{Q})}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = R^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla^{A\mathcal{P}^1(A\mathcal{Q})}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i_j} = \frac{\partial P^l_i}{\partial x^k} \frac{\partial}{\partial x^l_j}$, where

$$R^k_{l,ij} = \frac{\partial^2 P^k_l}{\partial x^j \partial x^i} + \frac{\partial^2 P^k_l}{\partial x^j \partial x^i}.$$

Using this we can give the following result:

Proposition 3.2.26. *Let $\mathcal{B}$ be a Cosserat continuum. $\mathcal{B}$ is locally homogeneous if, and only if, for each material particle $X \in \mathcal{B}$ there exists a local coordinate system $\left(x^i\right)$ over $\mathcal{U} \subseteq \mathcal{B}$ with $X \in \mathcal{U}$ such that the local non-holonomic covariant derivative of second order ∇ satisfies*

(i) $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = R^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i_j} = \frac{\partial P^l_i}{\partial x^k} \frac{\partial}{\partial x^l_j}$

where

$$R^k_{l,ij} = \frac{\partial^2 P^k_l}{\partial x^j \partial x^i} + \frac{\partial^2 P^k_l}{\partial x^j \partial x^i},$$

takes values in $\mathcal{D}\left(A\overline{\Omega}\left(\mathcal{B}\right)\right)$.

Let $\overline{\Delta} : T\mathcal{B} \to A\tilde{J}^1\left(F\mathcal{B}\right)$ be a linear section of $\sharp$ and $\nabla^{\overline{\Delta}}$ its associated covariant derivative. We can construct $\left(\overline{\Delta}_1\right)^1 \left(\overline{\Delta}_2\right)$ where

$$\overline{\Delta}_1 = A\overline{\Pi}^2_1 \circ \overline{\Delta} \quad ; \quad \overline{\Delta}_2 = A\tilde{\Pi}^2_1 \circ \overline{\Delta}.$$

Then, $\left(\overline{\Delta}_1\right)^1 \left(\overline{\Delta}_2\right)$ has an associated map $\nabla^{\left(\overline{\Delta}_1\right)^1\left(\overline{\Delta}_2\right)} : \mathfrak{X}\left(\mathcal{B}\right) \times \mathfrak{X}\left(F\mathcal{B}\right) \to \mathfrak{X}\left(F\mathcal{B}\right)$ which satisfies (i), (ii), (iii), (iv) and (v). So, $\overline{\Delta}$ is a prolongation if, and only if,

$$\nabla^{\overline{\Delta}} = \nabla^{\left(\overline{\Delta}_1\right)^1\left(\overline{\Delta}_2\right)}.$$

On the other hand, using that for all $\chi \in \mathfrak{X}\left(M\right)$, $\nabla^{\overline{\Delta}}_{\chi}$ is R^*_g-invariant we have that $\nabla^{\overline{\Delta}}_{\chi}$ preserves right-invariant vector fields on FM.
Then, we can project $\nabla^{\overline{\Delta}}$ onto a covariant derivative on M, ∇^1, in the

following way: Let χ, ς be two vector fields on M and ς^c the complete lift of ς over FM (see [15]). Then, ς^c is right-invariant which implies that $\nabla^{\overline{\Delta}}_{\chi}\varsigma^c$ is right-invariant. So, $\nabla^{\overline{\Delta}}_{\chi}\varsigma^c$ projects onto a unique vector field on M. This vector field will be $\nabla^1_{\chi}\varsigma$. It is straightforward to prove that ∇^1 is a covariant derivative over M; indeed, let $\left(x^i\right)$ be a local coordinate system on $\mathcal{B}$ such that

$$\overline{\Delta}\left(x^i, \frac{\partial}{\partial x^l}\right) = \left(x^i, \frac{\partial}{\partial x^l}, \Delta^j_{il}, \Delta^j_{,il}, \Delta^j_{i,kl}\right).$$

Then, ∇^1 satisfies that

$$\nabla^1_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Delta^k_{,ij} \frac{\partial}{\partial x^k}. \tag{3.56}$$

Hence, suppose that $\overline{\Delta}$ is a non-holonomic prolongation of second order. $\overline{\Delta}$ is an integrable prolongation if, and only if, ∇^1 is locally trivial, i.e., the Christoffel symbols are zero. There is an alternative way to construct ∇^1: Using the Lie algebroid morphism $\mathcal{D}$, we can construct a covariant derivative on M, $\nabla^{\overline{\Delta}_2}$, such that

$$\nabla^{\overline{\Delta}_2} = \nabla^1.$$

Remark 3.2.27. To summarize, we have introduced a new framework (groupoids and Lie algebroids) to study Cosserat media. In this arena, we have been able to express the homogeneity in several different (but equivalent) ways: Over the non-holonomic material groupoid of second order, over the associated Lie algebroid (which can be contructed without using the material groupoid) and over the Lie algebroid of derivations. Finally, using the interpretation over the algebroid of derivations, we have developed a method to know if a covariant derivative is a non-holonomic integrable prolongation without using coordinates.

Homogeneity with non-holonomic $\overline{G}-$structures of second order

Let us compare definition 2.1.23 of homogeneity respect to a reference crystal with our definition of homogeneity 3.2.5. It is important to recall

that the definition 3.2.5 of homogeneity does not depend on a reference crystal. So, these definitions cannot be equivalent (notice the difference with simple media 3.1). However, there are a close relation. In fact, $\mathcal{B}$ is homogeneous (resp. locally homogeneous) if, and only if, there exists a reference crystal $\overline{Z}_0^2$ such that $\mathcal{B}$ is homogeneous (resp. locally homogeneous) with respect to $\overline{Z}_0^2$.
To prove this, we will begin defining the following map

$$\begin{array}{rccc} \overline{\mathcal{G}}: & \Gamma\left(\overline{F}^2 M\right) & \to & \Gamma_{\left(\overline{\alpha},\overline{\beta}\right)}\left(\tilde{J}^1\left(FM\right)\right) \\ & \overline{P} & \mapsto & \overline{\mathcal{G}P}, \end{array}$$

such that

$$\overline{\mathcal{G}P}\left(x,y\right) = \overline{P}\left(y\right) \cdot \left[\overline{P}\left(x\right)\right]^{-1}, \ \forall x,y \in M,$$

where we are considering the equivalence class in $\tilde{J}^1\left(FM\right)$.

Remark 3.2.28. Notice that the following equalities relate $\overline{\Pi}_1^2$ (resp. $\tilde{\Pi}_1^2$) with $\overline{\rho}_1^2$ (resp. $\tilde{\rho}_1^2$)

(i) $\overline{\Pi}_1^2 \circ \overline{\mathcal{G}} = \mathcal{G} \circ \overline{\rho}_1^2$

(ii) $\tilde{\Pi}_1^2 \circ \overline{\mathcal{G}} = \mathcal{G} \circ \tilde{\rho}_1^2$

where $\mathcal{G} : \Gamma\left(FM\right) \to \Gamma_{\left(\alpha,\beta\right)}\left(\Pi^1\left(M,M\right)\right)$ has been defined in Eq. (3.13) [54] as follows:

$$\mathcal{G}P\left(x,y\right) = P\left(y\right) \cdot \left[P\left(x\right)\right]^{-1}, \ \forall x,y \in M.$$

Before working with second-order non-holonomic prolongations, we are interested in knowing when an element of $\Gamma_{\left(\overline{\alpha},\overline{\beta}\right)}\left(\tilde{J}^1\left(FM\right)\right)$ can be inverted by $\overline{\mathcal{G}}$. First, we consider $\overline{P} \in \Gamma\left(\overline{F}^2 M\right)$; then for all $x,y,z \in M$

$$\overline{\mathcal{G}P}\left(y,z\right) \cdot \overline{\mathcal{G}P}\left(x,y\right) = \overline{\mathcal{G}P}\left(x,z\right), \tag{3.57}$$

i.e., $\overline{\mathcal{G}P}$ is a morphism of Lie groupoids over the identity map on M from the pair groupoid $M \times M$ to $\tilde{J}^1\left(FM\right)$. Therefore, not all element of $\Gamma_{\left(\overline{\alpha},\overline{\beta}\right)}\left(\tilde{J}^1\left(FM\right)\right)$ can be inverted by $\overline{\mathcal{G}}$ but we can prove the following result:

Proposition 3.2.29. *Let $\overline{\mathcal{P}}$ be a section of $\tilde{J}^1(FM)$. There exists a section of $\overline{F}^2 M$ such that*

$$\overline{\mathcal{G}P} = \overline{\mathcal{P}}, \tag{3.58}$$

if, and only if, $\overline{\mathcal{P}}$ is a morphism of Lie groupoids over the identity map from the pair groupoid $M \times M$ to $\tilde{J}^1(FM)$.

Proof. We have already proved that Eq. (3.58) implies that $\overline{\mathcal{P}}$ is a morphism of Lie groupoids over the identity map from the pair groupoid $M \times M$ to $\tilde{J}^1(FM)$. Conversely, if Eq. (3.57) is satisfied we can define $\overline{P} \in \Gamma\left(\overline{F}^2 M\right)$ as follows

$$\overline{P}(x) = \overline{\mathcal{P}}(z_0, x) \cdot \overline{Z}_0^2,$$

where $\overline{Z}_0^2 \in \overline{F}^2 M$ with $\overline{\rho}^2\left(\overline{Z}_0^2\right) = z_0$ is fixed. Then, using Eq. (3.57), we have

$$\overline{\mathcal{G}P} = \overline{\mathcal{P}}.$$

□

However, there is not a unique $\overline{P}$ such that $\overline{\mathcal{G}P} = \overline{\mathcal{P}}$. In fact, let $\overline{P}$ and $\overline{Q}$ be sections of $\overline{F}^2 M$ such that

$$\overline{\mathcal{G}P} = \overline{\mathcal{G}Q}.$$

Then, there exists $\overline{g} \in \overline{G}^2(n)$ such that

$$\overline{P} = \overline{Q} \cdot \overline{g},$$

where we are choosing representatives of the equivalence class to do the jet composition (see remark 3.2.32).
Notice that the sections of $\left(\overline{\alpha}, \overline{\beta}\right)$ which are morphisms of Lie groupoids over the identity map from the pair groupoid $M \times M$ to $\tilde{J}^1(FM)$ are, precisely, the parallelisms.

Next, let us prove that $\overline{\mathcal{G}} : \Gamma\left(\overline{F}^2 M\right) \to \Gamma_{(\overline{\alpha},\overline{\beta})}\left(\tilde{J}^1(FM)\right)$ preserves prolongations. In fact, let $P^1(Q) \in \Gamma\left(\overline{F}^2 M\right)$ be a second-order non-holonomic prolongation of P and Q, then

$$\overline{\mathcal{G}}P^1(Q) = \mathcal{G}P^1(\mathcal{G}Q), \tag{3.59}$$

To prove the last equality, we use that

$$P^1(Q)(x) = j^1_{e_1,P(x)}\left(\overline{P \circ \psi_x}\right),$$

where $Q(x) = j^1_{0,x}\psi_x$ (see remark A.0.17). Then,

$$j^1_{0,y}\left(\overline{P \circ \psi_y}\right) \cdot j^1_{x,0}\left(\overline{P \circ \psi_x}\right)^{-1} = j^1_{x,y}\left(\overline{\mathcal{G}P \circ \left(\psi_y \circ \psi_x^{-1}\right)}\right),$$

and, as we know,

$$\mathcal{G}Q(x,y) = j^1_{x,y}\left(\psi_y \circ \psi_x^{-1}\right).$$

Then, taking into account that $\mathcal{G}$ preserves integrability (see section 3.1), we can assume that $\overline{\mathcal{G}}$ preserves integrable sections and integrable prolongations of $\overline{F}^2 M$.

Conversely, we want to study if we can invert integrable sections (resp. non-holonomic integrable prolongations of second-order) in $\tilde{J}^1(FM)$. Notice that both kinds of sections can be written as second-order non-holonomic prolongations and, in this way, we will study when we can invert non-holonomic prolongations of second-order.

So, let $\mathcal{P}^1(\mathcal{Q})$ be a second-order non-holonomic prolongation in $\tilde{J}^1(FM)$. Using Eq. (3.59) and remark 3.2.28, if we can invert $\mathcal{P}^1(\mathcal{Q})$ then, there exist $P, Q \in \Gamma(FM)$ such that

$$\overline{\mathcal{G}}P^1(Q) = \mathcal{P}^1(\mathcal{Q}).$$

Therefore, analogously to proposition 3.2.29, $\mathcal{P}$ and $\mathcal{Q}$ have to be Lie groupoid morphisms from the pair groupoid $M \times M$ to $\Pi^1(M,M)$.

Proposition 3.2.30. *Let $\mathcal{P}^1(\mathcal{Q})$ be a second-order non-holonomic prolongation in $\tilde{J}^1(FM)$. There exists a second-order non-holonomic prolongation in $\overline{F}^2 M$ such that*

$$\overline{\mathcal{G}}P^1(Q) = \mathcal{P}^1(\mathcal{Q}),$$

if, and only if, $\mathcal{P}$ and $\mathcal{Q}$ are morphisms of Lie groupoids from the pair groupoid $M \times M$ to $\Pi^1(M, M)$.

Now, notice that, by construction, every integral section of $\Pi^1(M, M)$ is a morphism of Lie groupoids from the pair groupoid $M \times M$ to $\Pi^1(M, M)$. So we can state the following result:

Corollary 3.2.31. *Let $\mathcal{P}^1(\mathcal{Q})$ be a second-order non-holonomic prolongation in $\tilde{J}^1(FM)$.*

(i) If $\mathcal{P}^1(\mathcal{Q})$ is integrable then, there exists an integrable section of $\overline{F}^2 M$, $P^1(Q)$, such that

$$\overline{\mathcal{G}}P^1(Q) = \mathcal{P}^1(\mathcal{Q}).$$

(ii) If $\mathcal{P}^1(\mathcal{Q})$ is a non-holonomic integrable prolongation of second order then, there exists a second-order non-holonomic integrable prolongation of $\overline{F}^2 M$, $P^1(Q)$, such that

$$\overline{\mathcal{G}}P^1(Q) = \mathcal{P}^1(\mathcal{Q}),$$

if, and only if, P is a morphism of Lie groupoids from the pair groupoid $M \times M$ to $\Pi^1(M, M)$.

This result could induce us to think that if $\overline{P}$ satisfies

$$\overline{\mathcal{G}P} = \mathcal{P}^1(\mathcal{Q}),$$

then, $\overline{P}$ is a second-order non-holonomic prolongation but this is not true and we will prove it in the following remark.

Remark 3.2.32. Let $\overline{P}$ and $\overline{Q}$ be sections of $\overline{F}^2 M$ such that

$$\overline{\mathcal{G}P} = \overline{\mathcal{G}Q}.$$

Then, for all $x, y \in M$,

$$\overline{P}(y) \cdot [\overline{P}(x)^{-1}] = \overline{Q}(y) \cdot [\overline{Q}(x)^{-1}],$$

i.e.,

$$[\overline{Q}(y)^{-1}] \cdot \overline{P}(y) = [\overline{Q}(x)^{-1}] \cdot \overline{P}(x), \tag{3.60}$$

where to do the above jet composition we are choosing a representative of the equivalence class. Thus, denoting $\overline{g} = [\overline{Q}(y)^{-1}] \cdot \overline{P}(y) \in \overline{G}^2(n)$ (which does not depend on the point because of Eq. (3.60)) we have

$$\overline{P} = \overline{Q} \cdot \overline{g}, \tag{3.61}$$

where, again, we are choosing representatives of the equivalence class to do the jet composition.
Conversely, if it satisfies Eq. (3.61) we immediately have that

$$\overline{\mathcal{G}P} = \overline{\mathcal{G}Q}.$$

Hence, we have shown that for each section of $\overline{F}^2 M$, $\overline{P}$,

$$\overline{\mathcal{G}}^{-1}\left(\overline{\mathcal{G}P}\right) = \{\overline{P} \cdot \overline{g} \ / \ \overline{g} \in \overline{G}^2(n)\} = \overline{P} \cdot \overline{G}^2(n).$$

Let $P^1(Q)$ be a second-order non-holonomic prolongation on $\overline{F}^2 M$ and $\overline{g} \in \overline{G}^2(n)$. Is $P^1(Q) \cdot \overline{g}$ a second-order non-holonomic prolongation? If the answer is negative, when can we ensure that $P^1(Q) \cdot \overline{g}$ is a second-order non-holonomic prolongation?, and, what about non-holonomic integrable prolongations of second order and integrable sections?
First, notice that

(i) $\overline{\rho}_1^2\left(P^1(Q)(x) \cdot \overline{g}\right) = P(x) \cdot \overline{\rho}_1^2(\overline{g})$.

(ii) $\tilde{\rho}_1^2\left(P^1(Q)(x) \cdot \overline{g}\right) = Q(x) \cdot \tilde{\rho}_1^2(\overline{g})$.

Then, denoting $\overline{\rho}_1^2(\overline{g})$ by g_1 and $\tilde{\rho}_1^2(\overline{g})$ by g_2, if $P^1(Q) \cdot \overline{g}$ is a non-holonomic prolongation of second order, it satisfies that

$$P^1(Q) \cdot \overline{g} = (P \cdot g_1)^1 (Q \cdot g_2).$$

So, let $\left(x^i\right)$ be a local coordinate system over a point $x \in M$ such that

$$P^1\left(Q\right)\left(x^i\right) = \left(\left(x^i, P^i_j\right), Q^i_j, Q^l_k \frac{\partial P^i_j}{\partial x^l}\right),$$

where

$$P\left(x^i\right) = \left(x^i, P^i_j\right), \qquad Q\left(x^i\right) = \left(x^i, Q^i_j\right).$$

Furthermore, using this coordinates, $\overline{g}$ can be seen

$$\overline{g} = \left(\left(0, A^i_j\right), A^i_{,j}, \alpha^i_{j,k}\right).$$

Then $P^1\left(Q\right) \cdot \overline{g}$ has the following local expression

$$\left(\left(x^i, P^i_l A^l_j\right), Q^i_l A^l_{,j}, P^i_r \alpha^r_{j,k} + Q^l_s \frac{\partial P^i_r}{\partial x^l} A^r_j A^s_{,k}\right).$$

On the other hand, the local expression of $\left(P \cdot g_1\right)^1\left(Q \cdot g_2\right)$ is the following

$$\left(\left(x^i, P^i_l A^l_j\right), Q^i_l A^l_{,j}, Q^l_s \frac{\partial P^i_r}{\partial x^l} A^r_j A^s_{,k}\right).$$

Hence, $P^1\left(Q\right) \cdot \overline{g}$ is a second-order non-holonomic prolongation if, and only if, $\alpha^i_{j,k} = 0$ for all i, j, k.

If we denote $\overline{Z}^2_0$ by $j_{e_1,A}\Psi \in \overline{G}^2\left(n\right)$, $P^1\left(Q\right)\overline{Z}^2_0$ is a second-order non-holonomic prolongation if, and only if, Ψ is locally constant at the matricial part, i.e.,

$$\Psi\left(x, Id\right) = \left(\psi\left(x\right), A\right), \ \forall x \in U.$$

Then, as a consequence, $P^1\left(Q\right)$ is a second-order non-holonomic integrable prolongation implies that $P^1\left(Q\right) \cdot \overline{g}$ is a second-order non-holonomic integrable prolongation if, and only if, $\alpha^i_{j,k} = 0$ for all i, j, k. On the other hand, $P^1\left(Q\right)$ is an integrable section implies that $P^1\left(Q\right) \cdot \overline{g}$ is an integrable section if, and only if, $\alpha^i_{j,k} = 0$ and $A^i_j = A^i_{,j}$ for all i, j, k.

So, in contrast with the simple media case, if $\overline{P} \in \overline{F}^2 M$ is such that $\overline{\mathcal{G}P}$ is a non-holonomic integrable prolongation of second order in $\tilde{J}^1(FM)$ (resp. integrable), then, we cannot ensure that $\overline{P}$ is a second-order non-holonomic integrable prolongation (resp. integrable) but there exists $\overline{g} \in \overline{G}^2(n)$ such that $\overline{P} \cdot \overline{g}$ is an integrable prolongation (resp. integrable) (see corollary 3.2.31).

Finally, as for the simple media case, we can generalize the map $\overline{\mathcal{G}}$ to a map which takes non-holonomic $\overline{G}-$structures of second order on M into reduced subgroupoids of $\tilde{J}^1(FM)$. Let $\overline{\omega}_{\overline{G}}(M)$ be a non-holonomic $\overline{G}-$structure of second order on M, then we consider the following set,

$$\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right) = \{L_y \cdot [L_x^{-1}] \ / \ L_x, L_y \in \overline{\omega}_{\overline{G}}(M)\},$$

where we are considering the equivalence class in $\tilde{J}^1(FM)$. We will denote $\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right)$ by $\tilde{J}^1_{\overline{G}}(FM)$. $\tilde{J}^1_{\overline{G}}(FM)$ is a reduced subgroupoid of $\tilde{J}^1(FM)$. In fact, taking a local section of $\overline{\omega}_{\overline{G}}(M)$,

$$\overline{P}_U : U \to \overline{\omega}_{\overline{G}}(U),$$

the map given by

$$\begin{array}{rccc} F_U : & \tilde{J}^1(FU) & \to & \overline{F}^2 U \times U \\ & \tilde{L}_{x,y} & \mapsto & \left(\tilde{L}_{x,y} \cdot [\overline{P}_U(x)], x\right) \end{array}$$

is a diffeomorphism which satisfies that $F_U\left(\tilde{J}^1_{\overline{G}}(FU)\right) = \overline{\omega}_{\overline{G}}(U) \times U$. Analogously to the case of parallelisms, we can prove that every reduced subgroupoid can be inverted by $\overline{\mathcal{G}}$ in a non-holonomic $\overline{G}-$structure of second order on M.
Fix $z_0 \in M$ and $\overline{Z}^2_0 \in \overline{F}^2 M$ with $\overline{\rho}^2\left(\overline{Z}^2_0\right) = z_0$. Then, we define

$$\overline{G} := \{\overline{Z}_0^{2^{-1}} \cdot \overline{g}_{z_0} \cdot \overline{Z}^2_0 \ / \ \overline{g}_{z_0} \in \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0}\} = \overline{Z}_0^{2^{-1}} \cdot \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0} \cdot \overline{Z}^2_0 \cong \left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0} \tag{3.62}$$

where $\left[\tilde{J}^1_{\overline{G}}\right]^{z_0}_{z_0}$ is the isotropy group of $\tilde{J}^1_{\overline{G}}(FM)$ over z_0. Therefore, $\overline{G}$ is clearly a Lie subgroup of $\overline{G}^2(n)$.
Then, we can generate a second-order non-holonomic $\overline{G}-$structure over M in the following way

$$\begin{aligned}\overline{\omega}_{\overline{G}}(M) &:= \{L_{z_0,x} \cdot \overline{Z}^2_0 \cdot \overline{g} \ / \ \overline{g} \in \overline{G}, \ L_{z_0,x} \in \tilde{J}^1_{\overline{G}}(FM)_{z_0}\} \\ &= \tilde{J}^1_{\overline{G}}(FM)_{z_0} \cdot \overline{Z}^2_0.\end{aligned}$$

Notice that $\tilde{J}^1_{\overline{G}}(FM)_{z_0}$ and $\overline{\omega}_{\overline{G}}(M)$ are clearly isomorphic.

Next, let $\overline{\omega}_{\overline{G}}(M)$ be an integrable second-order non-holonomic (resp. integrable prolongation) $\overline{G}-$structure on M. Using proposition A.0.14, proposition 3.2.11 and the fact of that $\overline{\mathcal{G}}$ preserves integrable sections (resp. integrable prolongations) we have that $\tilde{J}^1_{\overline{G}}(FM)$ is integrable (resp. an integrable prolongation).

Conversely, let $\tilde{J}^1_{\overline{G}}(FM)$ be an integrable (resp. integrable prolongation) Lie subroupoid of $\tilde{J}^1(FM)$. Then, we may construct an integrable (resp. integrable prolongation) second-order non-holonomic $\overline{G}-$structure $\overline{\omega}_{\overline{G}}(M)$ such that

$$\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right) = \tilde{J}^1_{\overline{G}}(FM). \tag{3.63}$$

To do this we just have to use proposition 3.2.10 (resp. proposition 3.2.18) and define it locally. However, not all second-order non-holonomic $\overline{G}-$structure wich satisfies Eq. (3.63) is integrable.

Remark 3.2.33. Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}-$structure on M and $\tilde{\omega}_{\tilde{G}}(M)$ be another second-order non-holonomic $\tilde{G}-$structure on M such that

$$\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right) = \overline{\mathcal{G}}\left(\tilde{\omega}_{\tilde{G}}(M)\right),$$

i.e., for all $j^1_{e_1,X}\Phi, j^1_{e_1,Y}\Theta \in \overline{\omega}_{\overline{G}}(M)$, there exist $j^1_{e_1,X}\tilde{\Phi}, j^1_{e_1,\tilde{Y}}\tilde{\Theta} \in \tilde{\omega}_{\tilde{G}}(M)$ such that

$$j^1_{X,Y}\left(\Theta \circ \Phi^{-1}\right) = j^1_{X,Y}\left(\tilde{\Theta} \circ \tilde{\Phi}^{-1}\right).$$

Then,

$$j^1_{e_1,A}\left(\tilde{\Theta}^{-1} \circ \Theta\right) = j^1_{e_1,A}\left(\tilde{\Phi}^{-1} \circ \Phi\right),$$

with $A \in Gl(n,\mathbb{R})$. Indeed, for all $j^1_{e_1,X'}\Psi \in \overline{\omega}_{\overline{G}}(M)$, there exists $j^1_{e_1,X'}\tilde{\Psi}, \in \tilde{\omega}_{\tilde{G}}(M)$ such that

$$j^1_{e_1,A}\left(\tilde{\Theta}^{-1} \circ \Theta\right) = j^1_{e_1,A}\left(\tilde{\Psi}^{-1} \circ \Psi\right),$$

So, denoting by $\overline{g} = j^1_{e_1,A}\left(\tilde{\Theta}^{-1} \circ \Theta\right) \in \overline{G}^2(n)$, we have

$$\overline{\omega}_{\overline{G}}(M) = \tilde{\omega}_{\tilde{G}}(M) \cdot \overline{g}, \tag{3.64}$$

where to do the above jet compositions we are choosing a representative of the equivalence class.
On the other hand, if Eq. (3.64) is satisfied it is obvious that

$$\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right) = \overline{\mathcal{G}}\left(\tilde{\omega}_{\tilde{G}}(M)\right).$$

Hence, we have shown that, for each non-holonomic $\overline{G}-$structure of second order $\overline{\omega}_{\overline{G}}(M)$ we have

$$\begin{aligned} \overline{\mathcal{G}}^{-1}\left(\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right)\right) &= \{\overline{\omega}_{\overline{G}}(M) \cdot \overline{g} \ / \ \overline{g} \in \overline{G}^2(n)\} \\ &= \overline{\omega}_{\overline{G}}(M) \cdot \overline{G}^2(n). \end{aligned}$$

Notice that Eq. (3.64) implies that the isotropy groups are conjugate, namely

$$\tilde{G} = \overline{g}^{-1} \cdot \overline{G} \cdot \overline{g}.$$

So, these structures are second-order non-holonomic conjugated $\overline{G}-$structures.
However, like in remark 3.2.32, not all second-order non-holonomic $\overline{G}-$structure $\overline{\omega}_{\overline{G}}(M)$ with $\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}}(M)\right)$ integrable (resp. integrable

prolongation) is integrable (resp. integrable prolongation) but, there exist $\overline{g} \in G^2(n)$ such that $\overline{\omega}_{\overline{G}}(M) \cdot \overline{g}$ is integrable (resp. integrable prolongation).

Now, we are in position to prove the above announced results. Let be $\mathcal{B}$ a Cosserat continuum and a crystal frame $\overline{Z}_0^2 \in \overline{F}^2\mathcal{B}$ at Z_0. Then, we have defined the second-order non-holonomic $\overline{G}_0$−structure of uniform references as follows

$$\overline{\omega}_{\overline{G}_0}(\mathcal{B}) = \overline{\Omega}_{z_0}(\mathcal{B}) \cdot \overline{Z}_0^2,$$

i.e.,

$$\overline{\mathcal{G}}\left(\overline{\omega}_{\overline{G}_0}(\mathcal{B})\right) = \overline{\Omega}(\mathcal{B}).$$

Therefore, there exists $\overline{g} \in \overline{G}^2(n)$ such that the second-order non-holonomic $\overline{G}_0$−structure $\overline{\omega}_{\overline{G}_0}(\mathcal{B}) \cdot \overline{g}$ is a second-order non-holonomic integrable prolongation if, and only if, $\overline{\Omega}(\mathcal{B})$ is a second-order non-holonomic integrable prolongation. So, using proposition 2.1.14 and proposition 3.2.23, we have the following result:

Proposition 3.2.34. *A Cosserat continuum $\mathcal{B}$ is homogeneous (resp. locally homogeneous) if, and only if, there exists a reference crystal $\overline{Z}_0^2$ such that $\mathcal{B}$ is homogeneous (resp. locally homogeneous) with respect to $\overline{Z}_0^2$.*

Hence, we our notion of homogeneity of a Cosserat medium $\mathcal{B}$ (which does not depend on a reference crystal) is equivalent to the existence of a configuration Φ such that $\mathcal{B}$ is homogeneous over the reference crystal $j^1_{e_1,Z_0}\Phi^{-1}$ (in terms of the non-holonomic $\overline{G}$−structures of second order).

Chapter 4

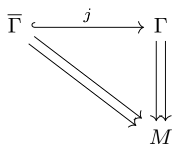
Characteristic distributions and material bodies

From the existence of structures of simple bodies $\mathcal{B}$ in which the material groupoid $\Omega(\mathcal{B})$ is not a Lie subgroupoid of the groupoid of $1-$jets $\Pi^1(\mathcal{B}, \mathcal{B})$ arises the need to develop more "*differentiable tools*". More generally, we will start studying the case of a general subgroupoid $\overline{\Gamma}$ of a Lie groupoid Γ to get results which may be applied to material bodies as well as other interesting examples.

4.1 Characteristic distribution

As we have said, sometimes it could be necessary to work with a groupoid which does not have a structure of Lie groupoid. In fact, the constitutive theory of continuum mechanics is an example (see section 4.2). In this case, the set of material isomorphisms has the structure of subgroupoid of a particular Lie groupoid: the $1-$jets groupoid on a manifold. However, this groupoid is not necessarily a LIE subgroupoid of the $1-$jets groupoid. This will be discussed in the next section in some detail.
In this section will work with a general subgroupoid of a given Lie

groupoid. Almost all the results are published in [51] (which is one the papers whose results are part of this thesis). However, there are here more results of what have been exposed in [51]. In fact, the development presented here is strictly more general due to corollary 4.1.4.
Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie subgroupoid of Γ) over the same manifold M. We will denote by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\epsilon}$ and $\overline{i}$ the restrictions of the structure maps of Γ to $\overline{\Gamma}$ (see the diagram below).



where j is the inclusion map. Now, we can construct a distribution $A\overline{\Gamma}^T$ over the manifold Γ in the following way,

$$g \in \Gamma \mapsto A\overline{\Gamma}^T_g \leq T_g\Gamma,$$

such that $A\overline{\Gamma}^T_g$ is the fibre of $A\overline{\Gamma}^T$ at g and it is generated by the (local) left-invariant vector fields $\Theta \in \mathfrak{X}_{loc}(\Gamma)$ whose flow at the identities is totally contained in $\overline{\Gamma}$, i.e.,

(i) Θ is tangent to the $\beta-$fibres,

$$\Theta(g) \in T_g\beta^{-1}(\beta(g)),$$

for all g in the domain of Θ.

(ii) Θ is invariant by left translations,

$$\Theta(g) = T_{\epsilon(\alpha(g))}L_g\left(\Theta\left(\epsilon\left(\alpha(g)\right)\right)\right),$$

for all g in the domain of Θ.

(iii) The (local) flow φ_t^{Θ} of Θ satisfies

$$\varphi_t^{\Theta}\left(\epsilon\left(x\right)\right) \in \overline{\Gamma},$$

for all $x \in M$.

Notice that, for each $g \in \Gamma$, the zero vector $0_g \in T_g\Gamma$ is contained in the fibre of the distribution at g, namely $A\overline{\Gamma}_g^T$ (we remit to the last section for non-trivial examples). On the other hand, it is easy to prove that a vector field Θ satisfies conditions (i) and (ii) if, and only if, its local flow φ_t^{Θ} is left-invariant or, equivalently,

$$L_g \circ \varphi_t^{\Theta} = \varphi_t^{\Theta} \circ L_g, \ \forall g, t.$$

Then, taking into account that all the identities are in $\overline{\Gamma}$ (because it is a subgroupoid of Γ), condition (iii) is equivalent to the following,

(iii)' The (local) flow φ_t^{Θ} of Θ at $\overline{g}$ is totally contained in $\overline{\Gamma}$, for all $\overline{g} \in \overline{\Gamma}$.

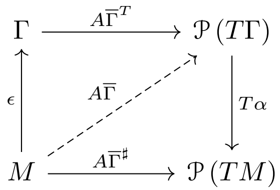
Thus, we are taking the left-invariant vector fields on Γ whose integral curves are confined inside or outside $\overline{\Gamma}$. It is also remarkable that, by definition, this distribution is differentiable. Remember that a distribution is differentiable (see appendix B) if for any point x and for any vector v_x of the distribution at x there exists a (local) vector field Θ tangent to the distribution such that,

$$\Theta\left(x\right) = v_x.$$

The distribution $A\overline{\Gamma}^T$ is called the *characteristic distribution of* $\overline{\Gamma}$. For the sake of simplicity, we will denote the family of the vector fields which satisfy conditions (i), (ii) and (iii) by $\mathcal{C}$. The local vector fields of $\mathcal{C}$ will be called *admissible vector fields*.

Remark 4.1.1. Our construction of the characteristic distribution associated to a subgroupoid $\overline{\Gamma}$ of a Lie groupoid Γ can be seen as a generalization of the construction of the associated Lie algebroid to a given Lie groupoid (see section 2.3). ◇

The structure of groupoid permits us to construct two more new objects associated to the distribution $A\overline{\Gamma}^T$. The first one is a smooth distribution over the base M denoted by $A\overline{\Gamma}^\sharp$. The second one is a "differentiable" correspondence $A\overline{\Gamma}$ which associates to any point x of M a vector subspace of $T_{\epsilon(x)}\Gamma$. Both constructions are characterized by the commutativity of the following diagram



where $\mathcal{P}(E)$ defines the power set of E. Therefore, for each $x \in M$, the fibres satisfy that

$$\begin{aligned} A\overline{\Gamma}_x &= A\overline{\Gamma}^T_{\epsilon(x)} \\ A\overline{\Gamma}^\sharp_x &= T_{\epsilon(x)}\alpha\left(A\overline{\Gamma}_x\right) \end{aligned}$$

The distribution $A\overline{\Gamma}^\sharp$ is called *base-characteristic distribution of* $\overline{\Gamma}$. It is remarkable that all the distributions introduced are not, necessarily, regular.

Notice that, taking into account that $A\overline{\Gamma}^T$ is locally generated by left-invariant vector field, we have that for each $g \in \Gamma$,

$$A\overline{\Gamma}^T_g = T_{\epsilon(\alpha(g))}L_g\left(A\overline{\Gamma}^T_{\epsilon(\alpha(g))}\right),$$

i.e., the characteristic distribution is *left-invariant.* In particular, the characteristic distribution and the base-characteristic distribution are characterized by the differentiable correspondence $A\overline{\Gamma}$ by using that

$$A\overline{\Gamma}^T_g = T_{\epsilon(\alpha(g))}L_g\left(A\overline{\Gamma}_{\alpha(g)}\right).$$

We could have used Grassmannian manifolds instead of power sets in the above diagram for the distributions but we prefer power sets because of the simplicity.

To summarize, associated to $\overline{\Gamma}$ we have three differentiable objects $A\overline{\Gamma}$, $A\overline{\Gamma}^T$ and $A\overline{\Gamma}^\sharp$. Now, we will study how these objects endow $\overline{\Gamma}$ with a sort of "differentiable" structure. Consider a left-invariant vector field Θ on Γ whose (local) flow φ_t^Θ at the identities is contained in $\overline{\Gamma}$. We want to prove that the characteristic distribution $A\overline{\Gamma}^T$ is invariant by the flow φ_t^Θ, i.e., for all $g \in \Gamma$ and t in the domain of φ_g^Θ we have

$$T_g\varphi_t^\Theta\left(A\overline{\Gamma}_g^T\right) = A\overline{\Gamma}^T_{\varphi_t^\Theta(g)}. \tag{4.1}$$

Indeed, let be $v_g = \Xi\left(g\right) \in A\overline{\Gamma}_g^T$ with $\Xi \in \mathcal{C}$. Then,

$$\begin{aligned} T_g\varphi_t^\Theta\left(v_g\right) &= T_g\varphi_t^\Theta\left(\Xi\left(g\right)\right) \\ &= \frac{\partial}{\partial s_{|0}}\left(\varphi_t^\Theta \circ \varphi_s^\Xi\left(g\right)\right), \end{aligned}$$

where φ_s^Ξ is the flow of Ξ.
Consider the (local) vector field Υ on Γ given by the pullback of Ξ by φ_t^Θ, i.e., for each h

$$\Upsilon\left(h\right) = \{\left(\varphi_t^\Theta\right)^*\Xi\}\left(h\right) = T_{\varphi_{-t}^\Theta(h)}\varphi_t^\Theta\left(\Xi\left(\varphi_{-t}^\Theta\left(h\right)\right)\right).$$

Then, the flow of Υ is $\varphi_t^\Theta \circ \varphi_s^\Xi \circ \varphi_{-t}^\Theta$ and, therefore, Υ is an admissible vector field. Furthermore,

$$T_g\varphi_t^\Theta\left(v_g\right) = \Upsilon\left(\varphi_t^\Theta\left(g\right)\right).$$

Hence, $T_g\varphi_t^\Theta\left(A\overline{\Gamma}_g^T\right) \subseteq A\overline{\Gamma}^T_{\varphi_t^\Theta(g)}$. We can prove the converse in an analogous way.
Thus, the characteristic distribution $A\overline{\Gamma}^T$ is locally generated by a family of vector fields $\mathcal{C}$, and it is invariant with respect this family. Remember

now the classical Stefan-Sussman's theorem B.0.17 which deals with the integrability of singular distributions (see appendix B for a detailed presentation of this result).

Theorem 4.1.2 (Stefan-Sussmann)**.** *Let D be a smooth singular distribution on a smooth manifold M. Then the following three conditions are equivalent:*

(a) D is integrable.

(b) D is generated by a family C of smooth vector fields, and is invariant with respect to C.

(c) D is the tangent distribution $D^{\mathcal{F}}$ of a smooth singular foliation $\mathcal{F}$.

There is still another theorem to deal with the integrability of generalized distributions which could be confused with the Stefan-Sussmann's theorem, the *Hermann's theorem* B.0.22, that states that any locally finitely generated differentiable involutive distribution on a manifold is integrable.
So, the distribution $A\overline{\Gamma}^T$ is the tangent distribution of a smooth singular foliation $\overline{\mathcal{F}}$. The leaf at a point $g \in \Gamma$ is denoted by $\overline{\mathcal{F}}(g)$. The collection of the leaves of $\overline{\mathcal{F}}$ at points of $\overline{\Gamma}$ is called the *characteristic foliation of* $\overline{\Gamma}$. Note that the leaves of the characteristic foliation covers $\overline{\Gamma}$ but it is not exactly a foliation of $\overline{\Gamma}$ (because $\overline{\Gamma}$ is not manifold).
The following assertions can be easily proved:

(i) For each $g \in \Gamma$,

$$\overline{\mathcal{F}}(g) \subseteq \Gamma^{\beta(g)}.$$

Indeed, if $g \in \overline{\Gamma}$, then

$$\overline{\mathcal{F}}(g) \subseteq \overline{\Gamma}^{\beta(g)}.$$

(ii) For each $g, h \in \Gamma$ such that $\alpha(g) = \beta(h)$, we have

$$\overline{\mathcal{F}}(g \cdot h) = g \cdot \overline{\mathcal{F}}(h).$$

The property (ii) is proved by arguments of maximility. On the other hand, the property (i) can be proved by checking the charts of the leaves given in the proof of the Stefan-Sussmann's theorem (see the proof of theorem B.0.17). It is remarkable that property (i) means that each leaf of the foliation $\overline{\mathcal{F}}$ which integrates $A\overline{\Gamma}^{T}$ is contained in just one $\beta-$fibre, i.e., for each $g \in \Gamma$ the leaf $\overline{\mathcal{F}}(g)$ satisfies that

$$\beta(h) = \beta(g),$$

for all $h \in \overline{\mathcal{F}}(g)$. Notice also that, one could expect that $\overline{\mathcal{F}}(g) = \overline{\Gamma}^{\beta(g)}$ but this is not true in general (see examples in section 4.2).
So, we have proved the following result.

Theorem 4.1.3. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie groupoid) over M. Then, there exists a foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.*

In this way, without assuming that $\overline{\Gamma}$ is a manifold, we prove that $\overline{\Gamma}$ is union of leaves of a foliation of Γ. This gives us some kind of "differentiable" structure over $\overline{\Gamma}$.
Let us consider a (local) left-invariant vector field $\Theta \in \mathcal{C}$. Then, the flow of Θ restrics to the fibers, i.e., Θ is a left-invariant vector field in Γ such that,

$$\Theta_{|\overline{\mathcal{F}}(g)} \in \mathfrak{X}\left(\overline{\mathcal{F}}(g)\right), \tag{4.2}$$

for all g in the domain of Θ. Reciprocally, left-invariant vector fields satisfying Eq. (4.2) are admissible vector fields.
It is important to note that we are working with the case in which $\overline{\Gamma}$ is a subgroupoid of Γ over the same manifold. In fact, we could do not have to impose it.

Corollary 4.1.4. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie groupoid). Then, there exists a maximal foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.*

Proof. Denote by N to the subset of M given by $\alpha\left(\overline{\Gamma}\right) = \beta\left(\overline{\Gamma}\right)$. Then, N is not necessarily a submanifold of M.
Let us now define the following subgroupoid $\overline{\Gamma}_M$ over M,

$$\overline{\Gamma}_M = \Gamma_{M-N} \sqcup \overline{\Gamma},$$

where Γ_{M-N} is the set of all the elements of Γ from points of $M - N$ to points of $M - N$ with $M - N$ as the collection of points of M outside N. Observe that all the identities at points of M are in $\overline{\Gamma}_M$. It is also remarkable that $\overline{\Gamma}_M$ is now a subgroupoid of Γ over the same manifold M. Then, we apply theorem 4.1.3 to $\overline{\Gamma}_M$ to get the characteristic foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}_M$ is a union of leaves of $\overline{\mathcal{F}}$.
Let $x \in M$ be a point which is not at N. Then,

$$\overline{\mathcal{F}}\left(\epsilon\left(x\right)\right) \subseteq {\overline{\Gamma}_M}^x \subseteq \Gamma_{M-N}.$$

Hence, the foliation $\overline{\mathcal{F}}$ of Γ satisfies that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$. □

Let $\overline{\Gamma}^2_M$ be another subgroupoid of Γ over M extending $\overline{\Gamma}$ such that there are not elements in $\overline{\Gamma}^2_M$ based at points of N which are not in $\overline{\Gamma}$, i.e., for all $g \in \overline{\Gamma}^2_M$ such that $\alpha\left(g\right) \in N$ it satisfies that $g \in \overline{\Gamma}$. Equivalently, there are not elements $g \in \overline{\Gamma}^2_M$ with $\beta\left(g\right) \in N$ outside $\overline{\Gamma}$. Notice that this is a natural imposition if we want that the characteristic foliation restricts to $\overline{\Gamma}$.
Then, there exists a set A of elemens of Γ from points of $M - N$ to points of $M - N$ containing the identities satisfying that

(i) $\overline{\Gamma}^2_M = A \sqcup \overline{\Gamma}$

(ii) $\epsilon\left(M - N\right) \subseteq A \subseteq \Gamma_{M-N}$

Denote by $\mathcal{C}^A_M$ and $\mathcal{C}_M$ to the admissible vector fields associated to $\overline{\Gamma}^2_M$ and $\overline{\Gamma}_M$ respectively. Then, by using **(i)** and **(ii)** we have that

$$\mathcal{C}^A_M \subseteq \mathcal{C}_M.$$

Therefore, all the leaves of the characteristic foliation associated to $\overline{\Gamma}^2_M$ are contained in the leaves of the characteristic foliation associated to $\overline{\Gamma}_M$. Then, the leaves associated to $\overline{\Gamma}_M$ are maximal and this is the reason because we chose the extension $\overline{\Gamma}_M$.
One could think that, maybe, the resulting leaves inside N do not depend on the choice of the subset A but this is not true (see example below).

Example 4.1.5. Consider $M = \mathbb{R}^2$, $\Gamma = \mathbb{R}^2 \times \mathbb{R}^2$ and $\overline{\Gamma} = \overline{B}_1(0) \times \overline{B}_1(0)$, where $\overline{B}_1(0)$ is the closed ball of centre 0 and radius 1 (what implies that $N = \alpha\left(\overline{\Gamma}\right) = \beta\left(\overline{\Gamma}\right) = \overline{B}_1(0)$). Then, consider

$$\overline{\Gamma}_M^2 = A \sqcup \overline{\Gamma},$$

with the set A defined as follows,

$$A = \epsilon\left(\overline{B}_1(0)^c\right) = \{(x, x) \;:\; x \notin \overline{B}_1(0)\},$$

with $\overline{B}_1(0)^c$ the complementary set of $\overline{B}_1(0)$ in $\mathbb{R}^2$.
So, for each point outside N we only add the identity at the point. Then, in this case, an admissible vector field has to be zero at points outside N. By continuity, the admisible vector fields are also necessarily zero at the boundary of $\overline{B}_1(0)$, i.e., at the sphere $S(1)$ of centre 0 and radius 1 in $\mathbb{R}^2$. On the other hand, inside the open ball $B_1(0)$ any local vector Θ field induce an admissible vector field as follows,

$$\Theta^l(x, y) = (\Theta(x), 0) \tag{4.3}$$

for all x, y in the domain of Θ. Therefore, the characteristic foliation $\overline{\mathcal{F}}_A$ of $\overline{\Gamma}_M^2$ is given by

(i) For all point $x \in S(1)$,

$$\overline{\mathcal{F}}_A(x, y) = \{x\} \times \{y\},$$

for any $y \in \mathbb{R}^2$.

(ii) For all point $x \in B_1(0)$,

$$\mathcal{F}_A(x, y) = B_1(0) \times \{y\},$$

for any $y \in \mathbb{R}^2$.

On the other hand, let us take the extension considered in the proof of corollary 4.1.4, i.e.,

$$\overline{\Gamma}_M = \left[\overline{B}_1(0)^c \times \overline{B}_1(0)^c\right] \sqcup \left[\overline{B}_1(0) \times \overline{B}_1(0)\right],$$

Then, taking into account that $\overline{B}_1(0)^c$ is an open subset of $\mathbb{R}^2$, any (local) vector field at $\overline{B}_1(0)^c$ induces an admissible vector field by Eq. (4.3). Analogously, any (local) vector field at $B_1(0)$ induces an admissible vector field for $\overline{\Gamma}_M$.
Let Θ be a (local) vector field with domain U such that $U \cap S(1) \neq \emptyset$. It is easy to see that if Θ is tangent to the sphere $S(1)$ then, Θ induces an admissible vector field for $\overline{\Gamma}_M$ by using Eq. (4.3). So, the characteristic distribution $A\overline{\Gamma}_M^T$ has dimesion larger than 1. On the other hand, there are vector fields which are not admissible vector fields, for instance,

$$\Theta(x, y) = \left(\frac{\partial}{\partial r^1}, 0\right),$$

where $\left(r^i\right)$ is the (global) canonical system of coordinates of $\mathbb{R}^2$, is not an admissible vector field. Therefore, necessarily,

$$dim\left(\left(A\overline{\Gamma}_M\right)^T_{(x,y)}\right) = 1.$$

for any $x \in S(1)$ and $y \in \mathbb{R}^2$. Thus, we have proved that the characteristic foliation $\overline{\mathcal{F}}$ associated to $\overline{\Gamma}_M$ is given by

(I) For all point $x \in S(1)$,

$$\overline{\mathcal{F}}(x, y) = S(1) \times \{y\},$$

for any $y \in \mathbb{R}^2$.

(II) For all point $x \in B_1(0)$,

$$\overline{\mathcal{F}}(x, y) = B_1(0) \times \{y\},$$

for any $y \in \mathbb{R}^2$.

In this way, these extensions ($\overline{\Gamma}_M$ and $\overline{\Gamma}^2_M$) generate strictly different characteristic leaves inside $\overline{\Gamma}$.

For a general subgroupoid $\overline{\Gamma}$ of a Lie groupoid Γ, we will call *characteristic distribution of* $\overline{\Gamma}$ to the characteristic distribution of $\overline{\Gamma}_M$ and it will be denoted by $A\overline{\Gamma}^T$. The collection of the leaves of $\overline{\mathcal{F}}$ at points of $\overline{\Gamma}$ is again called the *characteristic foliation of* $\overline{\Gamma}$.
As you could imagine, from the construction of the characteristic distribution, we obtain some condition of maximality.

Corollary 4.1.6. *Let $\overline{\mathcal{G}}$ be a left invariant foliation of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{G}}$. Then, the characteristic foliation $\overline{\mathcal{F}}$ is coarser that $\overline{\mathcal{G}}$, i.e.,*

$$\overline{\mathcal{G}}(g) \subseteq \overline{\mathcal{F}}(g), \ \forall g \in \Gamma. \tag{4.4}$$

Proof. Taking into account corollary 4.1.4 we may assume that $\overline{\Gamma}$ is a subgroupoid of Γ over the same manifold M. Let $\mathcal{D}$ be the family of (local) vector fields tangent to the foliation $\overline{\mathcal{G}}$. Then, the left-invariance of $\overline{\mathcal{G}}$ implies that any vector field $\Theta \in \mathcal{D}$ is tangent to the $\beta-$fibres. So, we may define a new left-invariant vector field Θ_L such that for each g,

$$\Theta_L(g) = T_{\epsilon(\alpha(g))} L_g \left(\Theta\left(\epsilon\left(\alpha\left(g\right)\right)\right)\right).$$

Denotes the family of left-invariant vector fields induced by the vector field in $\mathcal{D}$ by $\mathcal{D}_L$. Then, $\mathcal{D}_L$ generates the tangent distribution to $\overline{\mathcal{G}}$. In fact, for each $x \in M$, $T_{\epsilon(x)}\overline{\mathcal{G}}(\epsilon(x))$ is obviously generated by the evaluation of the vector fields of $\mathcal{D}_L$ at the identity $\epsilon(x)$ (the evaluation of the vector fields of $\mathcal{D}_L$ at the identity $\epsilon(x)$ results exactly in the vectors than the evaluation of the vector fields of $\mathcal{D}$ at the identity $\epsilon(x)$). Thus, the left-invariance of $\overline{\mathcal{G}}$ proves that $\mathcal{D}_L$ generates the tangent distribution to $\overline{\mathcal{G}}$.
Finally, using that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{G}}$ we have that $\mathcal{D}_L \subseteq \mathcal{C}$ and, therefore, Eq. (4.4) is satisfied.

□

Particularly, $\overline{\Gamma}^x$ *is a submanifold of* Γ *for all* $x \in M$ *if, and only if,* $\overline{\Gamma}^x = \overline{\mathcal{F}}(\epsilon(x))$ *for all* $x \in M$.

Notice that, in an analogous way to theorem 4.1.3, we can prove that the base-characteristic distribution $A\overline{\Gamma}^\sharp$ is also integrable. Thus, we will denote the resulting foliation which integrates the base-characteristic distribution over the base M by $\mathcal{F}$. For each point $x \in M$, the leaf of $\mathcal{F}$ through x will be denoted by $\mathcal{F}(x)$. $\mathcal{F}$ will be called the *base-characteristic foliation of* $\overline{\Gamma}$. Let us apply these results to a particular example. Let M be a manifold and $M \times M$ the pair groupoid (example 2.2.6). Then, any transitive subgroupoid of M is the pair groupoid $N \times N$ of a subset $N \subseteq M$. Then, using corollary 4.1.4 we have the following result.

Theorem 4.1.7. *Let M be a manifold and N be a subset of M. Then, there exists a maximal foliation $\mathcal{F}$ of M such that N is union of leaves.*

Proof. Let $N \times N \rightrightarrows N$ be the transitive pair groupoid of N. We will consider the subgroupoid

$$(N \times N)_M = [(M - N) \times (M - N)] \sqcup [N \times N] \rightrightarrows M,$$

of $M \times M \rightrightarrows M$. So, we may consider $\overline{\mathcal{F}}$ and $\mathcal{F}$ its characteristic foliation and base-characteristic foliation respectively.
Then, for each $x \in N$ we have that

$$\overline{\mathcal{F}}(x, x) \subseteq N \times \{x\}.$$

In fact, it satisfies that

$$\overline{\mathcal{F}}(x, x) = \mathcal{F}(x) \times \{x\}. \tag{4.5}$$

Hence, N is the union of the leaves of the base-characteristic foliation at points of N and we already have our foliation. □

Let us now explain the condition of maximality of the foliation. Let $\mathcal{G}$ be another foliation of M such that N is union of leaves. Then, for each $(x, y) \in M \times M$ we may define

$$\overline{\mathcal{G}}(x, y) = \mathcal{G}(x) \times \{y\}.$$

Hence, the family $\overline{\mathcal{G}} = \{\overline{\mathcal{G}}(x, y)\}_{(x,y) \in M \times M}$ defines a left invariant foliation of $M \times M$ such that $[(M - N) \times (M - N)]$ is union of leaves. Thus, the maximality condition of the characteristic foliation (corollary 4.1.6) implies that $\mathcal{G} \subset \mathcal{F}$, i.e., there is no another coarser foliation of M which divides N into union of leaves.

Notice that the maximal foliation given in theorem 4.1.7 permits us to endow N with differential structure which generalizes the structure of manifold. Indeed, N is a submanifold of M if, and only if, N consists of just one leaf of the foliation.
Let Θ be an admissible vector field of the subgroupoid $[(M-N)\times(M-N)]\sqcup[N\times N]\rightrightarrows M$. Then, the projection

$$\theta = T\alpha \circ \Theta \circ \epsilon,$$

on M is a vector field on M such that its flow at point of N is confined in N. Conversely, any vector fields θ whose flows at point of N is inside N may be lifted to an admissible vector field Θ by imposing that

$$\Theta\,(x,y) = (\theta\,(x)\,,0) \in T_xM \times T_yM,\ \forall x,y \in M. \tag{4.6}$$

Thus, the foliation given in the theorem 4.1.7 can be described by the vector fields on M whose flow at points of N is contained in N.

Example 4.1.8. Let $\sim$ be an equivalence relation on a manifold M, i.e., a binary relation that is reflexive, symmetric and transitive. Then, define the subset $\mathcal{O}$ of $M\times M$ given by

$$\mathcal{O} := \{(x,y)\ :\ x\sim y\}. \tag{4.7}$$

Then, $\mathcal{O}$ is a subgroupoid of $M\times M$ over M. In fact, this is equivalent to the properties reflexive, symmetric and transitive. For each $x\in M$, we denote by $\mathcal{O}_x$ to the orbit around x,

$$\mathcal{O}_x := \{y\ :\ x\sim y\}.$$

Notice that the orbits divide M into a disjoint union of subsets. However, these are not (necessarily) submanifolds.
On the other hand, the base-characteristic foliation gives us a foliation $\mathcal{F}$ of M such that

$$\mathcal{F}\,(x) \subseteq \mathcal{O}_x,\ \forall x\in M.$$

This foliation is maximal in the sense that there is no any other coarser foliation of M whose leaves are contained in the orbits (see theorem 4.1.13 and corollary 4.1.14).

Another example give rises to the so-called *material distributions*. This example will be presented in the next section.

Remark 4.1.9. We can construct another distribution $\mathcal{D}$ on $\overline{\Gamma}$ generated by the (local) vector fields whose flows are confined inside or outside $\overline{\Gamma}$. So, we will obtain a foliation $\overline{\mathcal{G}}$ of Γ such that $\overline{\Gamma}$ is covered by some of the leaves.
We could expect that the leaves at the identities $\overline{\mathcal{G}}\left(\epsilon\left(x\right)\right)$ are subgroupoids of Γ. However, this is not necessarily true. Because of this fact, we work with $A\overline{\Gamma}^{T}$ instead of $\mathcal{D}$ (see theorem 4.1.13).

◇

Next, we will prove that the leaves of $\mathcal{F}$ have even more geometric structure. In fact, we will find a Lie groupoid structure over each leaf of $\mathcal{F}$. To do this, we will prove the following technical proposition.

Proposition 4.1.10. *Let* $\Gamma \rightrightarrows M$ *be a Lie groupoid and* $\overline{\Gamma}$ *be a subgroupoid of* Γ *with* $\overline{\mathcal{F}}$ *and* $\mathcal{F}$ *the characteristic foliation and the base-characteristic foliation of* $\overline{\Gamma}$*, respectively. Then, for all* $x \in M$*, the mapping*

$$\alpha_{|\overline{\mathcal{F}}(\epsilon(x))} : \overline{\mathcal{F}}\left(\epsilon\left(x\right)\right) \to \mathcal{F}\left(x\right),$$

is a surjective submersion.

Proof. First, let us notice that

$$x \in \alpha\left(\overline{\mathcal{F}}\left(\epsilon\left(x\right)\right)\right) \cap \mathcal{F}\left(x\right) \neq \emptyset.$$

Next, consider a family $\{\Theta^{i}, \Xi^{j}\}_{i=1,\dots,r,j=1,\dots,s}$ of left-invariant vector fields in $\mathcal{C}$ such that $\{T_{\epsilon(x)}\alpha\left(\Theta^{i}\left(\epsilon\left(x\right)\right)\right)\}_{i=1,\dots,r}$ is a basis of $A\overline{\Gamma}^{\sharp}_{x}$ and $\{\Theta^{i}\left(\epsilon\left(x\right)\right), \Xi^{j}\left(\epsilon\left(x\right)\right)\}_{i=1,\dots,r,j=1,\dots,s}$ is a basis of $A\overline{\Gamma}^{T}_{\epsilon(x)}$.
Notice that the family $\{T\alpha \circ \Theta^{i} \circ \epsilon, T\alpha \circ \Xi^{j} \circ \epsilon\}_{i=1,\dots,r,j=1,\dots,s}$ of vector fields on M is tangent to the base-characteristic distribution $A\overline{\Gamma}^{\sharp}$. So, their flows at x are contained in $\mathcal{F}\left(x\right)$.
Furthermore, the map

$$\alpha \circ \varphi^{\Theta^{1}}_{t_{1}} \circ \epsilon \circ \cdots \circ \alpha \circ \varphi^{\Theta^{r}}_{t_{r}}\left(\epsilon\left(x\right)\right) = \alpha\left(\varphi^{\Theta^{1}}_{t_{1}} \circ \cdots \circ \varphi^{\Theta^{r}}_{t_{r}}\left(\epsilon\left(x\right)\right)\right),$$

defines a local chart of $\mathcal{F}(x)$ containing x, where $\varphi^{\Theta^i}_{t_i}$ is the (local) flow of Θ^i for each i. Following this argument, one can prove that $\alpha\left(\overline{\mathcal{F}}(\epsilon(x))\right)$ is an open subset of $\mathcal{F}(x)$.
Then, $\mathcal{F}(x)$ is the disjoint union of open subsets. Using that $\mathcal{F}(x)$ is connected we have that

$$\alpha\left(\overline{\mathcal{F}}(\epsilon(x))\right) = \mathcal{F}(x),$$

i.e., $\alpha_{|\overline{\mathcal{F}}(\epsilon(x))}$ is surjective. Hence, $\alpha_{|\overline{\mathcal{F}}(\epsilon(x))}$ is a submersion.

□

Let be $x \in M$ and $\Theta \in \mathfrak{X}(\mathcal{F}(x))$. Then, by using local sections of $\alpha_{|\overline{\mathcal{F}}(\epsilon(x))}$, we can extend (locally) Θ to a (left-invariant) vector field on $\overline{\mathcal{F}}(\epsilon(x))$. In this way, Θ is a local vector field tangent to the base-characteristic distribution if, and only if, it satisfies that

$$\Theta_{|\mathcal{F}(x)} \in \mathfrak{X}(\mathcal{F}(x)), \tag{4.8}$$

for all x in the domain of Θ.
As a corollary, we have the following interesting result.

Corollary 4.1.11. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ. Then, the manifolds $\overline{\mathcal{F}}(\epsilon(x)) \cap \alpha^{-1}(x)$ are Lie subgroups of Γ^x_x for all $x \in M$.*

Proof. Let be $h, g \in \overline{\mathcal{F}}(\epsilon(x)) \cap \alpha^{-1}(x)$. Then,

$$\overline{\mathcal{F}}(h \cdot g) = h \cdot \overline{\mathcal{F}}(g) = h \cdot \overline{\mathcal{F}}(\epsilon(x)) = \overline{\mathcal{F}}(h) = \overline{\mathcal{F}}(\epsilon(x)).$$

□

Another interesting consequence is that we can improve corollary 4.1.6

Corollary 4.1.12. *Let $\overline{\mathcal{G}}$ be a foliation of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{G}}$ and*

$$\overline{\mathcal{G}}(g) \subseteq \Gamma^{\beta(g)}, \ \forall g \in \Gamma.$$

Then, the characteristic foliation $\overline{\mathcal{F}}$ is coarser that $\overline{\mathcal{G}}$, i.e.,

$$\overline{\mathcal{G}}(g) \subseteq \overline{\mathcal{F}}(g), \ \forall g \in \Gamma.$$

Proof. Let us consider $\mathcal{D}$ as the family of (local) vector fields tangent to the foliation $\overline{\mathcal{G}}$. Fix $g \in \Gamma$ and $v_g \in T_g\overline{\mathcal{G}}(g)$. We may assume that there exists $\Theta \in \mathcal{D}$ such that

$$\Theta(g) = v_g. \tag{4.9}$$

By using proposition 4.1.10, we may have a local section σ_g of $\alpha_{|g \cdot \overline{\mathcal{F}}(\epsilon(\alpha(g)))} : g \cdot \overline{\mathcal{F}}(\epsilon(\alpha(g))) \to \mathcal{F}(\alpha(g))$ with $\sigma_g(\alpha(g)) = g$. So, we will define the following (local) left-invariant vector field Υ^{σ_g} on $g \cdot \overline{\mathcal{F}}(\epsilon(\alpha(g)))$ characterized by

$$\Upsilon^{\sigma_g}(\epsilon(y)) = T_{\sigma_g(y)} L_{\sigma_g(y)^{-1}}(\Theta(\sigma_g(y))) \tag{4.10}$$

Thus, the flow of Υ^{σ_g} is given by

$$\varphi_t^{\Upsilon^{\sigma_g}}(h) = h \cdot \left(\sigma_g(\alpha(h))^{-1}\right) \cdot \varphi_t^{\Theta}(\sigma_g(\alpha(h))).$$

Hence, Υ^{σ_g} generates an admissible vector field. Furthermore,

$$\Upsilon^{\sigma_g}(g) = \Theta(g) = v_g,$$

i.e., $v_g \in A\overline{\Gamma}_g^T$. □

Notice that, taking into account this result, we may "relax" conditions of the family of admissible vector fields. In fact, the characteristic distribution is generated by the (local) vector fields $\Theta \in \mathfrak{X}_{loc}(\Gamma)$ such that

(i) Θ is tangent to the $\beta-$fibres,

$$\Theta(g) \in T_g\Gamma^{\beta(g)},$$

for all g in the domain of Θ.

(ii) The (local) flow φ_t^{Θ} of Θ satisfies

$$\varphi_t^{\Theta}(\overline{g}) \in \overline{\Gamma},$$

for all $\overline{g} \in \overline{\Gamma}$.

Let us now construct an algebraic structure of a groupoid over the leaves of $\mathcal{F}$. We will consider the groupoid $\overline{\Gamma}(\mathcal{F}(x))$ generated by $\overline{\mathcal{F}}(\epsilon(x))$ by imposing that for all $\overline{g}, \overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$,

$$\overline{g}, \overline{g}^{-1}, \overline{h}^{-1} \cdot \overline{g} \in \overline{\Gamma}(\mathcal{F}(x)).$$

Notice that,

$$\overline{\mathcal{F}}(\epsilon(x)) = \overline{\mathcal{F}}\left(\overline{h}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right).$$

Therefore,

$$\overline{\mathcal{F}}\left(\overline{h}^{-1}\right) = \overline{h}^{-1} \cdot \overline{\mathcal{F}}(\epsilon(x)) = \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right).$$

On the other hand, let be $\overline{t} \in \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right)$. Then,

$$\overline{\mathcal{F}}\left(\overline{h} \cdot \overline{t}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\overline{t}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right) = \overline{\mathcal{F}}(\epsilon(x)).$$

i.e., $\overline{h} \cdot \overline{t} \in \overline{\mathcal{F}}(\epsilon(x))$ and, hence, $\overline{t}$ can be written as $\overline{h}^{-1} \cdot \overline{g}$ with $\overline{g} \in \overline{\mathcal{F}}(\epsilon(x))$. Thus, we have proved that

$$\overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right) \subset \overline{\Gamma}(\mathcal{F}(x)),$$

for all $\overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$. In fact, by following the same argument we have that

$$\overline{\Gamma}(\mathcal{F}(x)) = \sqcup_{\overline{g} \in \overline{\mathcal{F}}(\epsilon(x))} \overline{\mathcal{F}}(\epsilon(\alpha(\overline{g}))), \tag{4.11}$$

i.e., $\overline{\Gamma}(\mathcal{F}(x))$ can be depicted as a disjoint union of fibres at the identities. Let us now show that $\overline{\Gamma}(\mathcal{F}(x))$ is, in fact, a subgroupoid of $\overline{\Gamma}$. Consider two arbitrary elements $\overline{g}, \overline{h} \in \overline{\Gamma}(\mathcal{F}(x))$ with $\alpha\left(\overline{h}\right) = \beta(\overline{g})$. Then, we may assume that we are in one of the following options:

(i) $\overline{g}, \overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$. Then,

$$\begin{aligned} \overline{\mathcal{F}}\left(\overline{h} \cdot \overline{g}\right) &= \overline{h} \cdot \overline{\mathcal{F}}(\overline{g}) = \overline{h} \cdot \overline{\mathcal{F}}(\epsilon(x)) \\ &= \overline{\mathcal{F}}\left(\overline{h}\right) = \overline{\mathcal{F}}(\epsilon(x)) \end{aligned}$$

i.e., $\overline{h} \cdot \overline{g} \in \overline{\mathcal{F}}(\epsilon(x)) \subset \overline{\Gamma}(\mathcal{F}(x))$.

(ii) $\overline{g}^{-1}, \overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$. Then,

$$\begin{aligned}\overline{\mathcal{F}}\left(\overline{h}\cdot\overline{g}\right) &= \overline{h}\cdot\overline{\mathcal{F}}(\overline{g}) = \overline{h}\cdot\overline{\mathcal{F}}(\epsilon(\beta(\overline{g})))\\ &= \overline{\mathcal{F}}\left(\overline{h}\right) = \overline{\mathcal{F}}(\epsilon(x))\end{aligned}$$

So, $\overline{h}\cdot\overline{g} \in \overline{\mathcal{F}}(\epsilon(x)) \subset \overline{\Gamma}(\mathcal{F}(x))$.

(iii) $\overline{g}, \overline{h}^{-1} \in \overline{\mathcal{F}}(\epsilon(x))$,

$$\begin{aligned}\overline{\mathcal{F}}\left(\overline{h}\cdot\overline{g}\right) &= \overline{h}\cdot\overline{\mathcal{F}}(\overline{g}) = \overline{h}\cdot\overline{\mathcal{F}}(\epsilon(x))\\ &= \overline{\mathcal{F}}\left(\overline{h}\right) = \overline{\mathcal{F}}\left(\epsilon\left(\beta\left(\overline{h}\right)\right)\right)\end{aligned}$$

Hence, $\overline{h}\cdot\overline{g} \in \overline{\mathcal{F}}\left(\epsilon\left(\beta\left(\overline{h}\right)\right)\right) \subset \overline{\Gamma}(\mathcal{F}(x))$ (see Eq. (4.11)).

It is important to note that $\overline{\Gamma}(\mathcal{F}(x))$ may be equivalently defined as the smallest transitive subgroupoid of $\overline{\Gamma}$ which contains $\overline{\mathcal{F}}(\epsilon(x))$. Observe that the $\beta-$fibre of this groupoid at a point $y \in \mathcal{F}(x)$ is given by $\overline{\mathcal{F}}(\epsilon(y))$. Hence, the $\alpha-$fibre at y is

$$\overline{\mathcal{F}}^{-1}(\epsilon(y)) = i \circ \overline{\mathcal{F}}(\epsilon(y)).$$

Furthermore, the Lie groups $\overline{\mathcal{F}}(\epsilon(y)) \cap \Gamma_y$ are exactly the isotropy groups of $\overline{\Gamma}(\mathcal{F}(x))$. All these results imply the following one ([51]):

Theorem 4.1.13. *For each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}(\mathcal{F}(x))$ of Γ with base $\mathcal{F}(x)$.*

Proof. Let be $g \in \overline{\Gamma}(\mathcal{F}(x))$. Then, by proposition 4.1.10, the restriction

$$\beta_{|\overline{\mathcal{F}}^{-1}(g)} : \overline{\mathcal{F}}^{-1}\left(g^{-1}\right) \to \mathcal{F}(x), \tag{4.12}$$

is a surjective submersion, where $\overline{\mathcal{F}}^{-1}\left(g^{-1}\right) = i \circ \overline{\mathcal{F}}\left(g^{-1}\right)$. Using this fact, we will endow with a differentiable structure to $\overline{\Gamma}(\mathcal{F}(x))$. Let be

$g \in \overline{\Gamma}(\mathcal{F}(x))$. Consider $\sigma_g : U \to \overline{\mathcal{F}}^{-1}\left(g^{-1}\right)$ a (local) section of $\beta_{|\overline{\mathcal{F}}^{-1}\left(g^{-1}\right)}$ such that $\sigma_g(\beta(g)) = g$.
On the other hand, let $\{\Theta_i\}_{i=1}^{r}$ be a finite collection of vector fields in $\mathcal{C}$ such that $\{\Theta_i(\epsilon(\alpha(g)))\}_{i=1}^{r}$ is a basis of $A\overline{\Gamma}^{T}_{\epsilon(\alpha(g))}$. Then, a local chart $\varphi^{\Theta} : W \times U \to \Gamma$ over g can be given by

$$\varphi^{\Theta}(t_1, \ldots, t_r, z) = \sigma_g(z) \cdot [\varphi^{\Theta^r}_{t_r} \circ \cdots \circ \varphi^{\Theta^1}_{t_1}(\epsilon(\alpha(g)))]$$

where $\varphi^{\Theta^i}_{t}$ is the flow of Θ^i. By using that $\{\Theta^i(\epsilon(\alpha(g)))\}_{i=1}^{r}$ is a basis of $A\overline{\Gamma}^{T}_{\epsilon(\alpha(g))}$, we have that φ^{Θ} is an immersion. Also, it satisfies that

$$\varphi^{\Theta}(W \times U) \subseteq \overline{\Gamma}.$$

So, these charts give us an atlas over $\overline{\Gamma}(\mathcal{F}(x))$ which induces a Hausdorff second contable topology on $\overline{\Gamma}(\mathcal{F}(x))$ such that $\overline{\Gamma}(\mathcal{F}(x))$ is an immersed submanifold of Γ. To end the proof we just have to use Eq. (4.12) to prove that the source and the target mappings are submersions.

□

Thus, we have divided the manifold M into leaves $\mathcal{F}(X)$ which have a maximal structure of transitive Lie subgroupoids of Γ.

Corollary 4.1.14. *Let $\mathcal{G}$ be a foliation of M such that for each $x \in M$ there exists a transitive Lie subgroupoid $\Gamma(x)$ of Γ over the leaf $\mathcal{G}(x)$ contained in $\overline{\Gamma}$. Then, the base-characteristic foliation $\mathcal{F}$ is coarser than $\mathcal{G}$, i.e.,*

$$\mathcal{F}(x) \subseteq \mathcal{G}(x), \ \forall x \in M.$$

Futhermore it satisfies that

$$\Gamma(x) \subseteq \overline{\Gamma}(\mathcal{F}(x)).$$

Proof. Let $\mathcal{G}$ be a foliation of M in the condition of the corollary. Then, we consider the family of manifolds given by the $\beta-$fibres $\Gamma(x)^{x}$. By left translations we generate a foliation of Γ into submanifolds. By using corollary 4.1.12 we have finished.

□

As a particular consequence we have that: *$\overline{\Gamma}$ is a transitive Lie subgroupoid of Γ if, and only if, $M = \mathcal{F}(x)$ and $\overline{\Gamma} = \overline{\Gamma}(\mathcal{F}(x))$ for some $x \in M$.*
Let us give another consequence. Define the equivalence relation $\sim$ on M given by

$$x \sim y \quad \Leftrightarrow \quad \exists \overline{g} \in \overline{\Gamma}, \ \alpha(\overline{g}) = x, \ \beta(\overline{g}) = y.$$

Then, example 4.1.8 provides another foliation $\mathcal{G}$ of M.

Corollary 4.1.15. *The base-characteristic foliation $\mathcal{F}$ based on the groupoid $\overline{\Gamma}$ is contained in the foliation $\mathcal{G}$ based on the equivalence relation $\sim$.*

Proof. For each $x \in M$, it satisfies that $\mathcal{F}(x) \times \mathcal{F}(x)$ defines a transitive Lie subgroupoid of $M \times M$ over $\mathcal{F}(x)$. So, using corollary 4.1.14 we have done. $\square$

Notice that the main difference between the foliation $\mathcal{G}$ and $\mathcal{F}$ is that, with $\mathcal{F}$, we are not only requesting "*regularity*" on the base manifold M but on the groupoid $\overline{\Gamma}$. In particular, assume that $\overline{\Gamma}$ is a transitive subgroupoid of Γ. Then, $\mathcal{G}$ consists in one unique leaf equal to M. However, if $\overline{\Gamma}$ is not a Lie subgroupoid of Γ the characteristic foliation $\overline{\mathcal{F}}$ is not given by the $\overline{\beta}-$fibres and, hence, the base-characteristic foliation $\mathcal{F}$ does not have (necessarily) one unique leaf equal to M.

4.2 Uniformity and homogeneity

In this section we will apply the results of the previous section to the case of simple materials. Particularly, let $\mathcal{B}$ be a simple body with $W : \Pi^1(\mathcal{B}, \mathcal{B}) \to V$ as the mechanical response (see section 2.1). Then, we may define the so-called material groupoid $\Omega(\mathcal{B})$ (see **Prelude** 2.3) which is a subgroupoid of the groupoid of $1-$jets $\Pi^1(\mathcal{B}, \mathcal{B})$. So, it makes sense to apply here the development of the section 4.1. The development of this section is summarized in the published articles [39, 53] which are part of the thesis.
Let Θ be an admissible left-invariant vector field on $\Pi^1(\mathcal{B}, \mathcal{B})$, i.e.,

$\varphi_t^{\Theta}(\epsilon(X)) \in \Omega(\mathcal{B})$ for all $X \in \mathcal{B}$. Then, for all $g \in \Pi^1(\mathcal{B},\mathcal{B})$, we have that

$$\begin{aligned} TW(\Theta(g)) &= \frac{\partial}{\partial t_{|0}}\left(W\left(\varphi_t^{\Theta}(g)\right)\right) \\ &= \frac{\partial}{\partial t_{|0}}\left(W\left(g\cdot\varphi_t^{\Theta}(\epsilon(\alpha(g)))\right)\right) \\ &= \frac{\partial}{\partial t_{|0}}(W(g)) = 0. \end{aligned}$$

Therefore, analogously to the case of the material algebroid (see Eq. (3.12) in section 3.1), we have that

$$TW(\Theta) = 0 \tag{4.13}$$

The converse is proved in the same way.
So, the characteristic distribution $A\Omega(\mathcal{B})^T$ of the material groupoid is generated by the (left-invariant) vector fields on $\Pi^1(\mathcal{B},\mathcal{B})$ which are in the kernel of TW. This characteristic distribution will be called *material distribution*. The base-characteristic distribution $A\Omega(\mathcal{B})^\sharp$ will be called *body-material distribution*. Let us recall that the left-invariant vector fields on $\Pi^1(\mathcal{B},\mathcal{B})$ which satisfy Eq. (4.13) are called admissible vector fields and the family of these vector fields is denoted by $\mathcal{C}$.
Denote by $\overline{\mathcal{F}}(\epsilon(X))$ and $\mathcal{F}(X)$ the foliations associated to the material distribution and the body-material distribution respectively. For each $X \in \mathcal{B}$, we will denote the Lie groupoid $\Omega(\mathcal{B})(\mathcal{F}(X))$ by $\Omega(\mathcal{F}(X))$ (see theorem 4.1.13).

Graded uniformity

Notice that, strictly speaking, in continuum mechanics a *sub-body* of a body $\mathcal{B}$ is an open submanifold of $\mathcal{B}$ but, here, the foliation $\mathcal{F}$ gives us submanifolds of different dimensions. So, we will consider a more general definition so that, a *material submanifold (or generalized sub-body) of* $\mathcal{B}$ is just a submanifold of $\mathcal{B}$. A generalized sub-body $\mathcal{P}$ inherits certain material structure from $\mathcal{B}$. In fact, we will measure the material response of a material submanifold $\mathcal{P}$ by restricting W to the $1-$jets of local

diffeomorphisms ϕ on $\mathcal{B}$ from $\mathcal{P}$ to $\mathcal{P}$. However, it is easy to observe that a material submanifold of a body is not exactly a body. See [50] for a discussion on this subject.
Then, as a corollary of theorem 4.1.3 and corollary 4.1.14, we have the following result.

Theorem 4.2.1. *For all $X \in \mathcal{B}$, $\Omega(\mathcal{F}(X))$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. Thus, any body $\mathcal{B}$ can be covered by a maximal foliation of smoothly uniform material submanifolds.*

Notice that, in this case "maximal" means that any other foliation $\mathcal{H}$ by smoothly uniform material submanifolds is thinner than $\mathcal{F}$, i.e.,

$$\mathcal{H}(X) \subseteq \mathcal{F}(X), \ \forall X \in \mathcal{B}.$$

So, by using the material distributions we have been able to prove a very intuitive result: Let $\mathcal{B}$ a general (smoothly uniform or not) simple material. Then, $\mathcal{B}$ may be decomposed into "*smoothly uniform parts*" and this decomposition is, in fact, a foliation of the material body.
We can ask now the same question for (not generally smooth) uniformity. To solve this problem, we will take advantage of the development made in example 4.1.8. In fact, consider a simple body $\mathcal{B}$ and define the equivalence relation $\sim$ on $\mathcal{B}$ given by

$$X \sim Y \quad \Leftrightarrow \quad \exists j^1_{X,Y}\psi \in \Omega(\mathcal{B}).$$

Thus, two material particles are related if, and only if, they are materially isomorphic. Doing this, we are playing down what happens with the set of material isomorphisms, we are only worried about the points on the body. Thus, example 4.1.8 provides another foliation $\mathcal{G}$ of M.
Therefore, by using corollary 4.1.14, we obtain the similar result which we were looking for.

Theorem 4.2.2. *Any simple body $\mathcal{B}$ can be covered by a maximal foliation of uniform material submanifolds.*

In this case the maximality has the same meaning and the foliation will be denoted by $\mathcal{G}$. Notice that corollary 4.1.15 provides a very intuitive results in this context: *The uniform leaves are generally bigger that smoothly uniform leaves.*

Remark 4.2.3. Imagine that there is, at least, a $1-$jet $g \in \Omega^X(\mathcal{B})$ for some $X \in \mathcal{B}$ such that

$$g \notin \overline{\mathcal{F}}(\epsilon(X)).$$

Then, we are not including g inside any of the transitivie Lie subgroupoids $\Omega(\mathcal{F}(X))$. Thus, these material isomorphisms are being discarded. Nevertheless

$$\overline{\mathcal{F}}(g) = g \cdot \overline{\mathcal{F}}(\epsilon(\alpha(g))), \tag{4.14}$$

and, indeed, $\overline{\mathcal{F}}(\epsilon(\alpha(g)))$ is contained in $\Omega(\mathcal{F}(\alpha(g)))$, i.e., using Eq. (4.14), we can reconstruct $\overline{\mathcal{F}}(g)$. ◇

Finally, using the body-material distribution, we will be able to define a more general notion of smooth uniformity. This notion was introduced in [39]. We will end up using the foliation by uniform subbodies to interpret it over the material groupoid.

Definition 4.2.4. Let be a body $\mathcal{B}$ and a body point $X \in \mathcal{B}$. Then, $\mathcal{B}$ is said to be *uniform of grade p at X* if $A\Omega(\mathcal{B})^\sharp_X$ has dimension p. $\mathcal{B}$ is *uniform of grade p* if it is uniform of grade p at all the points.

Note that, smooth uniformity is a particular case of graded uniformity. In fact, $\mathcal{B}$ is smoothly uniform if, and only if, $\mathcal{B}$ is uniform of grade 3. Equivalently, $\mathcal{B}$ is uniform of grade 3 if, and only if, $A\Omega(\mathcal{B})^\sharp_X$ has dimension 3 for all $X \in \mathcal{B}$, i.e., there exists just one leaf of the material foliation equal to $\mathcal{B}$. Hence, the material groupoid $\Omega(\mathcal{B})$ is a Lie subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$ whose $\overline{\beta}-$fibres integrate the material distribution.

Corollary 4.2.5. *Let be a body $\mathcal{B}$ and let $X \in \mathcal{B}$ be a body point. $\mathcal{B}$ is uniform of grade p at X if, and only if, the uniform leaf $\mathcal{F}(X)$ at X has dimension p.*

Corollary 4.2.6. *Let $\mathcal{B}$ be a body. $\mathcal{B}$ is uniform of grade p if, and only if, the body-material foliation is regular of rank p.*

It is important to highlight again that the body-material foliation has certain condition of maximality (see theorem 4.2.1). In fact, suppose that there exists another foliation $\mathcal{H}$ of $\mathcal{B}$ by smoothly uniform material submanifolds. Then, for all $X \in \mathcal{B}$ we have that

$$\mathcal{H}(X) \subseteq \mathcal{F}(X), \ \forall X \in \mathcal{B}.$$

So, we have the following results:

Corollary 4.2.7. *Let be a body $\mathcal{B}$ and let $X \in \mathcal{B}$. $\mathcal{B}$ is uniform of grade greater or equal to p at X if, and only if, there exists a foliation $\mathcal{H}$ of $\mathcal{B}$ by smoothly uniform submanifolds such that the leaf $\mathcal{H}(X)$ at X has dimension greater or equal to p.*

Corollary 4.2.8. *Let $\mathcal{B}$ be a body. $\mathcal{B}$ is uniform of grade p if, and only if, the body can be foliated by smoothly uniform material submanifolds of dimension p.*

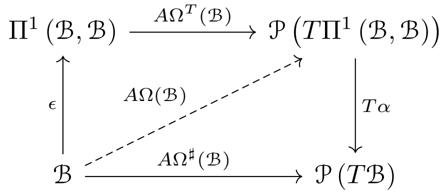
Homogeneity

This section is devoted to deal we the definition of homogeneity. As we already know, a body is (locally) homogeneous if it admits a (local) configuration ϕ which induces a left (local) smooth field of material isomorphisms $\mathcal{P}$ given by

$$\mathcal{P}(Y, X) = j^1_{Y,X}\left(\phi^{-1} \circ \tau_{\phi(X)-\phi(Y)} \circ \phi\right), \tag{4.15}$$

for all body point Y in the domain of ϕ and a fixed $X \in \mathcal{B}$ (see definition 2.1.10). Roughly speaking, a body is said to be homogeneous if we can choose a section of the material groupoid which is constant on the body. As we have said before, local homogeneity is clearly more restrictive than smooth uniformity. In fact, in this case, the smooth fields of material isomorphisms (see definition 2.3.47) are induced by particular (local) configurations.
However, in a purely intuitive picture, homogeneity can be interpreted as the absence of defects. So, it makes sense to develop a concept of some kind of homogeneity for non-uniform materials which measures the absence of defects and generalizes the known one. In the literature we can already find some partial answer of this question ([9, 35] for FGM's and [34, 39] for laminated and bundle materials).

Recall that the material distributions are characterized by the commutativity of the following diagram



As we have proved in the previous section, the body-material foliation $\mathcal{F}$ divides the body into smoothly uniform components.
Let us now provide the intuition behind the definition of homogeneity of a non-uniform body. A non-uniform body will be *(locally) homogeneous* when each smoothly uniform material submanifold $\mathcal{F}(X)$ is (locally) homogeneous and all the uniform material submanifolds can be straightened at the same time.
Thus, we need to clarify what we understand by homogeneity of submanifolds of $\mathcal{B}$.

Definition 4.2.9. *Let $\mathcal{B}$ be a simple body and $\mathcal{N}$ be a submanifold of $\mathcal{B}$. $\mathcal{N}$ is said to be homogeneous if, and only if, for all point $X \in \mathcal{N}$ there exists a local configuration ψ of $\mathcal{B}$ on an open subset $U \subseteq \mathcal{B}$, with $\mathcal{N} \subseteq U$, which satisfies that*

$$j^1_{Y,Z}\left(\psi^{-1} \circ \tau_{\psi(Z)-\psi(Y)} \circ \psi\right),$$

is a material isomorphism for all $Y, Z \in \mathcal{N}$. We will say that $\mathcal{N}$ is locally homogeneous if there exists a covering of $\mathcal{N}$ by open subsets U_a of $\mathcal{B}$ such that $U_a \cap \mathcal{N}$ are homogeneous submanifolds of $\mathcal{B}$. $\mathcal{N}$ is said to be (locally) inhomogeneous if it is not (locally) homogeneous.

Notice that, the definitions of homogeneity and local homogeneity for smoothly uniform materials (definition 2.1.10) are generalized by this one whether $\mathcal{N} = \mathcal{B}$ or $\mathcal{N}$ is just an open subset of $\mathcal{B}$.

Now, taking into account that $\mathcal{F} = \{\mathcal{F}(X)\}_{X\in\mathcal{B}}$ is a foliation, there is a kind of compatible atlas which is called a foliated atlas (see appendix B). In fact, $\{\left(\left(x^i_a\right), U_a\right)\}_a$ is a foliated atlas of $\mathcal{B}$ associated to $\mathcal{F}$ whenever for each $X \in U_a \subseteq \mathcal{B}$ we have that $U_a := \{-\epsilon < x^1_a < \epsilon, \dots, -\epsilon < x^3_a < \epsilon\}$

for some $\epsilon > 0$, such that the $k-$dimensional disk $\{x_a^{k+1} = \ldots = x_a^3 = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}(X)$ with U_a which contains X, and each $k-$dimensional disk $\{x_a^{k+1} = c_{k+1}, \ldots, x_a^3 = c_3\}$, where $c_{k+1}, \ldots, c_3$ are constants, is wholly contained in some leaf of $\mathcal{F}$. Intuitively, this atlas straightens (locally) the partition $\mathcal{F}$ of $\mathcal{B}$.
The existence of these kinds of atlases and the maximality condition over the smoothly uniform material submanifolds $\mathcal{F}(X)$ induces us to give the following definition.

Definition 4.2.10. *Let $\mathcal{B}$ be a simple body. $\mathcal{B}$ is said to be locally homogeneous if, and only if, for all point $X \in \mathcal{B}$ there exists a local configuration ψ of $\mathcal{B}$, with $X \in U$, which is a foliated chart and it satisfies that*

$$j^1_{Y,Z}\left(\psi^{-1} \circ \tau_{\psi(Z)-\psi(Y)} \circ \psi\right),$$

is a material isomorphism for all $Z \in U \cap \mathcal{F}(Y)$. We will say that $\mathcal{B}$ is homogeneous if $U = \mathcal{B}$. The body $\mathcal{B}$ is said to be (locally) inhomogeneous if it is not (locally) homogeneous.

It is remarkable that, as we have said above, all the uniform leaves $\mathcal{F}(X)$ of an homogeneous body are homogeneous. Therefore, the definition of homogeneity for a smoothly uniform body coincides with definition 2.1.10. Notice also that, the condition that all the leaves $\mathcal{F}(X)$ are homogeneous is not enough in order to have the homogeneity of the body $\mathcal{B}$ because there is also a condition of compatibility with the foliation structure of $\mathcal{F}$.

Let us recall a result presented in section 2.1 given in [31] (see also [32] or [92]) which characterizes the homogeneity by using $G-$structures.
Fix $\overline{g}_0$ be a frame at $Z_0 \in \mathcal{B}$. Then, assuming that $\mathcal{B}$ is smoothly uniform, the set

$$\Omega(\mathcal{B})_{Z_0} \cdot \overline{g}_0 := \{\overline{g} \cdot \overline{g}_0 \ : \ \overline{g} \in \Omega(\mathcal{B})_{Z_0}\},$$

where $\cdot$ defines the composition of $1-$jets, is a G_0-structure over $\mathcal{B}$ where

$$G_0 := \overline{Z_0}^{-1} \cdot G(Z_0) \cdot \overline{Z_0}.$$

Proposition 4.2.11. *Let be a frame* $\overline{g}_0 \in F\mathcal{B}$. *If* $\mathcal{B}$ *is homogeneous then the* G_0-*structure given by* $\Omega(\mathcal{B}) \cdot \overline{g}_0$ *is integrable. Conversely,* $\Omega(\mathcal{B}) \cdot \overline{g}_0$ *is integrable implies that* $\mathcal{B}$ *is locally homogeneous.*

Thus, the next step will be to give a similar result for this generalized homogeneity. Because of the lack of uniformity we have to use groupoids instead of $G-$structures.

Let $\mathbb{S} := \{\mathbb{S}(x) \ : \ x \in \mathbb{R}^n\}$ be a canonical foliation of $\mathbb{R}^n$ (see example B.0.6), i.e., for all $x = \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n$ the leaf $\mathbb{S}(x)$ at x

$$\mathbb{S}(x) := \{\left(y^1, \ldots, y^p, x^{p+1}, \ldots, x^n\right) \ : \ y^i \in \mathbb{R}, \ i = 1, \ldots, p\},$$

for some $1 \leq p \leq n$.
Remember that for any foliation $\mathcal{G}$ on a manifold Q there exists a map

$$dim_{\mathcal{G}} : Q \to \{0, \ldots, dim(Q)\},$$

such that for all $x \in Q$

$$dim_{\mathcal{G}}(x) = dim(\mathcal{G}(x)).$$

It is important to remark that in the case of $\mathbb{S}$ the dimension $dim_{\mathbb{S}}$ characterizes the foliation $\mathbb{S}$. Thus, with abuse of notation, we could say that the map $dim_{\mathbb{S}}$ is the foliation.

Let $\mathbb{S}$ be a canonical foliation of $\mathbb{R}^n$ with dimension $p = dim_{\mathbb{S}}$. Then, as a generalization of the frame bundle of $\mathbb{R}^n$, we define the $p-$*graded frame groupoid* as the following subgroupoid of $\Pi^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$\Pi^1_p(\mathbb{R}^n, \mathbb{R}^n) = \{j^1_{x,y}\psi \in \Pi^1(\mathbb{R}^n, \mathbb{R}^n) \ : \ y \in \mathbb{S}(x)\}.$$

Notice that the restriction of $\Pi^1_p(\mathbb{R}^n, \mathbb{R}^n)$ to any leaf $\mathbb{S}(x)$ is a transtive Lie subgroupoid of $\Pi^1(\mathbb{R}^n, \mathbb{R}^n)$ with all the isotropy groups isomorphic to $Gl(n, \mathbb{R})$. However, the groupoid $\Pi^1_p(\mathbb{R}^n, \mathbb{R}^n)$ is not necessarily a Lie subgroupoid of $\Pi^1(\mathbb{R}^n, \mathbb{R}^n)$. In fact, $\Pi^1_p(\mathbb{R}^n, \mathbb{R}^n)$ *is a Lie subgroupoid of* $\Pi^1(\mathbb{R}^n, \mathbb{R}^n)$ *if, and only if,* $\mathbb{S}$ *is regular foliation.*
A *standard flat* $G-$*reduction of grade* p is a subgroupoid $\Pi^1_{G,p}(\mathbb{R}^n, \mathbb{R}^n)$

of $\Pi^1_p(\mathbb{R}^n, \mathbb{R}^n)$ such that the restrictions $\Pi^1_{G,p}(\mathbb{S}(x), \mathbb{S}(x))$ to the leaves $\mathbb{S}(x)$ are transitive Lie subgroupoids of $\Pi^1(\mathbb{R}^n, \mathbb{R}^n)$ on the leaf $\mathbb{S}(x)$. It is remarkble that in this case all the isotropy groups of $\Pi^1_{G,p}(\mathbb{S}(x), \mathbb{S}(x))$ are conjugated.
Clearly, all the structures introduced in this section can be restricted to any open subset of $\mathbb{R}^n$.
Let $\psi : U \to \overline{U}$ be a (local) configuration on $U \subseteq \mathcal{B}$. Then, ψ induces a Lie-groupoid isomorphism,

$$\begin{array}{rccl} \Pi\psi : & \Pi^1(U,U) & \to & \Pi^1\left(\overline{U},\overline{U}\right) \\ & j^1_{X,Y}\phi & \mapsto & j^1_{\psi(X),\psi(Y)}\left(\psi \circ \phi \circ \psi^{-1}\right) \end{array}$$

Proposition 4.2.12. *Let $\mathcal{B}$ be a simple body. If $\mathcal{B}$ is homogeneous the material groupoid is isomorphic (via a global configuration) to a standard flat $G-$reduction. Conversely, if the material groupoid is isomorphic (via a local configuration) to a standard flat $G-$reduction, then $\mathcal{B}$ is locally homogeneous.*

Notice that, in the context of principal bundles, a $G-$structure is integrable if, and only if, there exists a local configuration which induces an isomorphism from the $G-$structure to a standard flat $G-$structure. Observe also that this results in a natural generalization of the proposition 3.1.5. In fact, implicitly, we are generalizing the notion of integrability 3.1.1.

Finally, we will use the material distribution to give another characterization of homogeneity.
Let $\mathcal{B}$ be a homogeneous body with $\psi = \left(x^i\right)$ as a (local) homogeneous configuration. Then, by using the fact that ψ is a foliated chart, we have that the partial derivatives are tangent to $A\Omega(\mathcal{B})^\sharp$, i.e., for each $X \in U$

$$\frac{\partial}{\partial x^l}_{|X} \in A\Omega(\mathcal{B})^\sharp_X ,$$

for all $1 \leq l \leq dim(\mathcal{F}(X)) = K$. Thus, there are local functions $\Lambda^j_{i,j}$ such that for each $l \leq K$ the (local) left-invariant vector field on $\Pi^1(\mathcal{B}, \mathcal{B})$ given by

$$\frac{\partial}{\partial x^l} + \Lambda^j_{i,l}\frac{\partial}{\partial y^j_i},$$

is tangent to $A\Omega(\mathcal{B})^T$, where $\left(x^i, y^j, y^j_i\right)$ are the induced coordinates of $\left(x^i\right)$ in $\Pi^1(\mathcal{B},\mathcal{B})$. Equivalently, the local functions $\Lambda^j_{i,l}$ satisfy that

$$\frac{\partial W}{\partial x^l} + \Lambda^j_{i,l}\frac{\partial W}{\partial y^j_i} = 0,$$

for all $1 \leq l \leq K$. Next, since for each two points $X, Y \in U$ the $1-$jet given by $j^1_{X,Y}\left(\psi^{-1} \circ \tau_{\psi(Y)-\psi(X)} \circ \psi\right)$ is a material isomorphism, we can choose $\Lambda^j_{i,l} = 0$.

Proposition 4.2.13. *Let $\mathcal{B}$ be a simple body. $\mathcal{B}$ is homogeneous if, and only if, for each $X \in \mathcal{B}$ there exists a local chart $\left(x^i\right)$ on $\mathcal{B}$ at X such that,*

$$\frac{\partial W}{\partial x^l} = 0, \tag{4.16}$$

for all $l \leq dim\left(\mathcal{F}(X)\right)$.

Notice that Eq. (4.16) implies that the partial derivatives of the coordinates $\left(x^i\right)$ up to $dim\left(\mathcal{F}(X)\right)$ are tangent to the material distribution and, therefore, the coordinates are foliated. So, Eq. (4.16) gives us an apparently more straightforward way to express this "*generalized*" homogeneity.

4.3 Examples

We will devote this section to apply the notions of graded uniformity 4.2.4 and homogeneity 4.2.10 for non-uniform bodies. In particular, we will present two (family of) examples: an homogeneous non-uniform body and an (generally) inhomogeneous non-uniform body.

We will see that, in some of them, the material groupoid is not a Lie groupoid (this kind of examples justify the study of groupoids without structure of Lie groupoids). We shall also give the decomposition of the material by smoothly uniform material submanifolds provided by the characteristic distribution. In these examples we will also show that the leaves $\overline{\mathcal{F}}\left(\epsilon(X)\right)$ are contained in the $\overline{\beta}-$fibres of $\Omega(\mathcal{B})$ but they do not coincide in general.

Example 1

Let $\mathcal{B}$ be a simple material body for which there exists a reference configuration ψ_0 from $\mathcal{B}$ to the 3−dimensional open ball $B_r(0)$ of centre 0 and radius $r > 1$ in $\mathbb{R}^3$ which induces the following mechanical response

$$\begin{array}{rlcl} W: & \Pi^1(B_r(0), B_r(0)) & \to & \mathfrak{gl}(3,\mathbb{R}) \\ & j^1_{X,Y}\phi & \mapsto & f\left(\| X \|^2\right)\left(F \cdot F^T - I\right), \end{array}$$

such that

$$f(t) = \begin{cases} 1 & if \quad t \leq 1 \\ 1 + e^{-\frac{1}{t-1}} & if \quad t > 1 \end{cases}$$

where $\mathfrak{gl}(3,\mathbb{R})$ is the algebra of matrices, F is the Jacobian matrix of ϕ at X with respect to the canonical basis of $\mathbb{R}^3$ and I is the identity matrix. Here, the (global) canonical coordinates of $\mathbb{R}^3$ are denoted by $\left(X^I\right)$ and $X = \left(X^1, X^2, X^3\right)$ with respect to these coordinates. In these coordinates, we allow the summation convention to be in force regardless of the placement of the indices. We also identify the coordinate system in the spatial configuration with that of the reference configuration.

Notice that f is constant up to 1 and strictly increasing thereafter. For this reason, one can immediately conclude that $B_r(0)$ is not uniform. In fact, there are no material isomorphisms joining any two points X and Y such that

$$f\left(\| X \|^2\right) \neq f\left(\| Y \|^2\right).$$

So, let us study the derivatives of W in order to find the grades of uniformity of the points of the body $B_r(0)$. We obtain

$$\frac{\partial W^{ij}}{\partial F^k_M} = f\left(\| X \|^2\right)\left[\delta^i_k\, F^j_M + \delta^j_k\, F^i_M\right]$$

$$\frac{\partial W^{ij}}{\partial X^I} = 2X^I \frac{\partial f}{\partial t}\left(F^i_K F^j_K - \delta^{ij}\right)$$

We are looking for left-invariant (local) vector fields Θ on $\Pi^1(B_r(0), B_r(0))$ satisfying

$$\Theta\left(W^{ij}\right) = 0. \tag{4.17}$$

Let $\left(X^I, Y^I, F^i_J\right)$ be the induced coordinates of $\left(X^I\right)$ on $\Pi^1(B_r(0), B_r(0))$. A left-invariant vector field Θ can be expressed as follows,

$$\Theta\left(X^I, Y^J, F^j_I\right) = \left(\left(X^I, Y^J, F^j_I\right), \delta X^I, 0, F^j_L \delta P^L_I\right).$$

Hence, Θ satisfies Eq. (4.17) if, and only if,

$$\begin{aligned}\Theta\left(W^{ij}\right) &= f\left(\| X \|^2\right)\left(F^i_L F^j_M + F^j_L F^i_M\right)\delta P^L_M \\ &+ 2X^I \delta X^I \frac{\partial f}{\partial t}\left(F^i_K F^j_K - \delta^{ij}\right) = 0.\end{aligned}$$

Let us focus first on the open given by the restriction $\| X \|^2 < 1$. Then, the above equation turns into the following,

$$\left(F^i_L F^j_M + F^j_L F^i_M\right)\delta P^L_M = 0 \quad \forall i, j = 1, 2, 3 \tag{4.18}$$

for every Jacobian matrix $F = \left(F^j_L\right)$ of a local diffeomorphism ϕ on $B_r(0)$. Since the bracketed expression is symmetric in L and M for every i and j, it follows that δP is a skew-symmetric matrix. We remark that this condition does not impose any restriction on the components δX^I of the admissible vector fields on the base vectors $\partial / \partial X^I$. In other words, any family of local functions $\{\delta X^I, \delta P^L_M\}$ on the open restriction $\{\| X \|^2 < 1\}$ of the body $B_r(0)$, such that $\delta P = \left(\delta P^L_M\right)$ is a skew-symmetric matrix, generates a vector field

$$\Theta\left(X^I, Y^J, F^j_I\right) = \left(\left(X^I, Y^J, F^j_I\right), \delta X^I, 0, F^j_L \delta P^L_I\right),$$

which satisfies Eq. (4.17). It follows that the body characteristic distribution of the sub-body $B_1(0)$ is a regular distribution of dimension 3. Therefore, this sub-body is uniform, as one would expect from the constancy of the function f thereat. Note also that the part lost when projecting the characteristic distribution onto the body, namely the skew-symmetric matrices δP, consists precisely of the Lie algebra of the orthogonal group. This is nothing but the manifestation of the fact that our sub-body is isotropic.

Next we will study the open subset of $B_r(0)$ such that $\| X \|^2 > 1$. For this case, Eq. (4.17) is satisfied if, and only if,

$$\begin{aligned} & f\left(\| X \|^2\right)\left(F^i_L F^j_M + F^j_L F^i_M\right)\delta P^L_M + 2X^I \delta X^I \frac{\partial f}{\partial t} F^i_K F^j_K = \\ = \quad & 2X^I \delta X^I \frac{\partial f}{\partial t}\delta^{ij} \end{aligned}$$

The function on the left-hand side of this equation is homogeneous of degree 2 with respect to the matrix coordinate F, but the function on the right-hand side does not depend on F. Consequently, the above equation can be identically satisfied if, and only if,

$$\delta X^I \frac{\partial f}{\partial t} = 0, \ \forall I \tag{4.19}$$

Notice that, the map f is strictly monotonic (and, hence, a submersion) at the open subset given by the condition $\| X \|^2 > 1$. Then, for any point X in this open subset we have that

$$T_X f^{-1}\left(f\left(\| X \|^2\right)\right) = Ker\left(T_X f\right),$$

i.e., the tangent space of the level set $f^{-1}\left(f\left(\| X \|^2\right)\right)$, which is the sphere $\| Y \|^2 = \| X \|^2$, consists of vectors $V = \left(V^1, V^2, V^3\right)$ such that

$$V^1 \frac{\partial f}{\partial t_{|\| X \|^2}} = 0.$$

In this way, a vector field Θ satisfies Eq. (4.17) if, and only if, δP is skew-symmetric and the proyection $T\alpha \circ \Theta \circ \epsilon$ is tangent to the vertical spheres $\| Y \|^2 = C$. Therefore, for each point $X = \left(X^1, X^2, X^3\right)$ with

$X^1 > 0$, the uniform leaf is given by the sphere $\| Y \|^2 = \| X \|^2$. As a consequence, the uniform leaf at the points satisfying $\| X \|^2 = 1$ is, again, the sphere $\| Y \|^2 = 1$.
We conclude that the body is uniform of grade 3 for all points $X = \left(X^1, X^2, X^3\right) \in B_r(0)$ such that $\| X \|^2 < 1$, and it is uniform of grade 2 otherwise. It should be remarked that the sphere $\| X \|^2 = 1$ is uniform of grade 2, even though its points are materially isomorphic to those in the subset with $\| X \|^2 < 1$ (see figure 4.1).

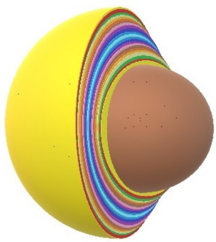


Figure 4.1: Material foliation of $B_r(0)$

Finally, the material body $B_r(0)$ is locally homogeneous. In fact, let us consider the spherical coordinates (r, θ, φ) of $B_r(0)$. Then,

$$\frac{\partial W^{ij}}{\partial \theta} = \frac{\partial W^{ij}}{\partial \varphi} = 0,$$

i.e., by using proposition 4.2.13, $B_r(0)$ is homogeneous and the coordinates (r, θ, φ) are homogeneous coordinates.

Example 2

We will consider a perturbation of the model introduced by Coleman [14] and Wang [91] called *simple liquid crystal* introduced in section 2.3. This kind of materials could be called *laminated simple liquid crystals*.

In this case we will consider a simple body $\mathcal{B}$ together with a reference configuration ψ_1 from $\mathcal{B}$ the open ball $\mathcal{B}_1 = B_r(0)$ in $\mathbb{R}^3$ of radius r and center $0 \in \mathbb{R}^3$. Furthermore, ψ_1 induces on $\mathcal{B}_1$ a mechanical response $\mathcal{W}$ determined by the following objects:

(i) A fixed vector field e on $\mathcal{B}_1$ such that $e(X) \neq 0$ for all $X \in \mathcal{B}_1$.

(ii) Two differentiable maps $r, J : \Pi^1(\mathcal{B}_1, \mathcal{B}_1) \to \mathbb{R}$ in the following way

- $r\left(j^1_{X,Y}\phi\right) = g(Y)\left(T_X\phi(e(X)), T_X\phi(e(X))\right) + \parallel X \parallel^2$
- $J\left(j^1_{X,Y}\phi\right) = det(F)$

where F is the Jacobian matrix of ϕ with respect to the canonical basis of $\mathbb{R}^3$ at X, g is a Riemannian metric on $\mathcal{B}_1$ and $\parallel \cdot \parallel$ the euclidean norm of $\mathbb{R}^3$.

(iii) A differentiable map $\widehat{W} : \mathbb{R}^2 \to V$, with V a finite-dimensional $\mathbb{R}-$vector space.

Thus, these three objects induce a structure of simple body by considering the mechanical response $\mathcal{W} : \Pi^1(\mathcal{B}_1, \mathcal{B}_1) \to V$ as the composition

$$\mathcal{W} = \widehat{W} \circ (r, J).$$

Let us now fix the canonical (global) coordinates $\left(X^I\right)$ of $\mathbb{R}^3$. Then, these coordinates induce a (canonic) isomorphism $T\mathcal{B}_1 \cong \mathcal{B}_1 \times \mathbb{R}^3$. By using this isomorphism any vector $V_X \in T_X\mathcal{B}_1$ can be equivalently expressed as $\left(X, V^i\right)$ in $\mathcal{B}_1 \times \mathbb{R}^3$. For the same reason, r can be written as follows:

$$r\left(j^1_{X,Y}\phi\right) = g(Y)\left(F^j_L e^L(X), F^j_L e^L(X)\right) + \parallel X \parallel^2,$$

where $e(X) = \left(X, e^I(X)\right) \in \mathcal{B}_1 \times \mathbb{R}^3$. Both expressions will be used with the same notation as long as there is no confusion.

Now, we want to study the conditions characterizing the material distribution $A\Omega^T(\mathcal{B}_1)$. In particular, we should study the admissible left-invariant vector fields Θ on $\Pi^1(\mathcal{B}_1, \mathcal{B}_1)$, i.e.,

$$\Theta(\mathcal{W}) = 0. \tag{4.20}$$

Notice that, for each $U = \left(U_i^j\right) \in gl(3, \mathbb{R})$ and $v = \left(v^i\right) \in \mathbb{R}^3$ we have that,

(i)
$$\frac{\partial r}{\partial X_{|j^1_{X,Y}\phi}}(v) =$$
$$= 2\ g(Y)\left(F^j_L e^L(X), F^j_L \frac{\partial e^L}{\partial X^M_{|X}} v^M\right) +\ 2\ v^L X^L$$

(ii)
$$\frac{\partial r}{\partial F_{|j^1_{X,Y}\phi}}(U) = 2\ g(Y)\left(F^j_L e^L(X), U^j_L e^L(X)\right)$$

(iii)
$$\frac{\partial J}{\partial F_{|j^1_{X,Y}\phi}}(U) = det(F)\, Tr\left(F^{-1} \cdot U\right)$$

We are denoting the coordinate $X^K(X)$ by X^K.
Let $\left(X^I, Y^J, Y^J_I\right)$ be the induced local coordinates of the canonical coordinates $\left(X^I\right)$ of $\mathbb{R}^3$ in $\Pi^1(\mathcal{B}_1, \mathcal{B}_1)$. Then, Θ can be expressed as follows,

$$\Theta\left(X^I, Y^J, F^j_I\right) = \left(\left(X^I, Y^J, F^j_I\right), \delta X^I, 0, F^j_L \delta P^L_I\right).$$

Hence, Θ is an admissible vector field if, and only if,

$$
\begin{aligned}
0 \quad = \quad & 2\frac{\partial \widehat{W}}{\partial r_{|j^1_{X,Y}\phi}} g\left(Y\right)\left(F^j_L e^L\left(X\right), F^j_L \frac{\partial e^L}{\partial X^M_{|X}} \delta X^M\left(X\right)\right) \\
& + 2\ \frac{\partial \widehat{W}}{\partial r_{|j^1_{X,Y}\phi}} \delta X^L\left(X\right) X^L \\
& + 2\ \frac{\partial \widehat{W}}{\partial r_{|j^1_{X,Y}\phi}} g\left(Y\right)\left(F^j_L e^L\left(X\right), F^j_L \delta P^L_M\left(X\right) e^M\left(X\right)\right) \\
& + det\left(F\right)\frac{\partial \widehat{W}}{\partial J_{|j^1_{X,Y}\phi}} Tr\left(\delta P^j_I\left(X\right)\right),
\end{aligned}
$$

for all $j^1_{X,Y}\phi \in \Pi^1\left(\mathcal{B}_1, \mathcal{B}_1\right)$. So, a sufficient but not necessary condition would be that, for each $1-$jet of local diffeomorphisms $j^1_{X,Y}\phi$ on $\mathcal{B}_1$, it satisfies

(1) $Tr\left(\delta P^j_I\left(X\right)\right) = 0$

(2) Denoting $\mathcal{L}^L_X = \dfrac{\partial e^L}{\partial X^M_{|X}} \delta X^M\left(X\right) + \delta P^L_M\left(X\right) e^M\left(X\right)$, then

$$g\left(Y\right)\left(F^j_L e^L\left(X\right), F^j_L\left(\mathcal{L}^L_X\right)\right) = -\delta X^L\left(X\right) X^L$$

In order to turns this conditions into necessary conditions we will assume that $\widehat{W}$ is an immersion and, hence, **(1)** and **(2)** are equivalent to the above equation.

In this way, $\mathcal{B}_1$ is smoothly uniform if, and only if, for each vector V_X at X there exists a family of local functions $\{\delta X^I, \delta P^i_J\}$ at X satisfying **(1)**, **(2)** and

$$\delta X^I\left(X\right) = V^I,\ \forall i$$

where $V_X = \left(X, V^I\right) \in \mathcal{B}_1 \times \mathbb{R}^3$.

Let us focus on the second condition: Suppose that $\left\langle V^I, X\right\rangle \neq 0$. Then, fixing the spatial point $X \in \mathcal{B}_1$ the map depending on the matrix coordinates F_i^j,

$$g(Y)\left(F_L^j e^L(X), F_L^j\left(\frac{\partial e^L}{\partial X^M_{|X}} V^M + \delta P^l_M(X) e^M(X)\right)\right) \tag{4.21}$$

is equal to $-\delta X^L(X) X^L$ which does not depend on the matrix coordinates F_i^j and it is not zero. However, the map (4.21) depends bilinearly on F_I^j. So, (4.21) cannot be constant (respect to F_i^j) and different from zero at the same time. Therefore, we could conclude that $\mathcal{B}_1$ is not smoothly uniform.
This fact opens the possibility of studying the graduated uniformity of these materials. Notice that, as we have proved, any admissible vector field Θ satisfies that

$$\delta X^L(X) X^L = 0, \tag{4.22}$$

where $\Theta\left(X^I, Y^J, F_I^j\right) = \left(\left(X^I, Y^J, F_I^j\right), \delta X, 0, F_L^j \delta P_I^L\right)$ respect to the coordinates $\left(X^I, Y^J, F_I^j\right)$ on $\Pi^1(\mathcal{B}_1, \mathcal{B}_1)$.
Let $X \in \mathcal{B}_1$ be a point of the body different to 0. Then, the map given by $\| \cdot \|^2$ restricted to $\mathcal{B}_1$ has full rank at X. In fact, the level set of $\| \cdot \|^2$ at $\| X \|^2$ is given by the sphere $S(\| X \|)$ of radius $\| X \|$ and centre 0 and it satisfies that

$$T_X S(\| X \|) = Ker\left(T_X \| \cdot \|^2\right).$$

So, the tangent space of the sphere $S(\| X \|)$ at X consists of the vectors $V_X = \left(X, V^I\right) \in \mathcal{B}_1 \times \mathbb{R}^3$ satisfying

$$V^L X^L = 0 \tag{4.23}$$

Then, any vector V_X satisfying Eq. (4.23) can be expanded by a (local) vector field θ^S on $S(\| X \|)$ such that

$$\theta^S(X) = V_X.$$

It is an easy exercise to prove that θ^S can be extended to a vector field θ on an open neighbourhood U of $\mathcal{B}_1$ which is tangent to all the spheres intersecting U. Then, expressing θ in the canonical coordinates $\left(X^I\right)$ as follows

$$\theta\left(X^I\right) = \left(X^I, \delta X^I\right),$$

the functions δX^i satisfy Eq. (4.22). Therefore, by using the non-degeneracy of the Riemannian metric g, it is enough to realize that there exist infinite families of local maps δP_I^j at X from the body to $\mathbb{R}$ satisfying that

(1)" $\delta P_I^i = 0.$

(2)" $\dfrac{\partial e^J}{\partial X^L}\delta X^L = -\delta P_L^j e^L, \ \forall j,$

Therefore, the local vector fields given by,

$$\Theta\left(X^I, Y^J, F_I^j\right) = \left(\left(X^I, Y^J, F_I^j\right), \delta X^I, 0, F_L^j \delta P_I^L\right),$$

satisfy Eq. (4.20) and $\delta X^I(X) = V_X$. Then, we have already proved that the grade of uniformity of any point X at $\mathcal{B}_1$ different from 0 is 2 and the smoothly uniform submanifolds are given by the spheres $S(\| X \|)$. Then, obviously, the grade of uniformity of 0 is 0 and the smoothly uniform submanifold at 0 is $\{0\}$. Therefore, ignoring the origin point, $\mathcal{B}_1$ is a "laminated" body covered by smoothly uniform submanifolds of dimension 2 with a kind of structure similar to *liquid crystals*. Notice that the picture of this material is similar to the previous one (figure 4.1) but the "*solid core*" is just a point. However, in this case, the homogeneity is not ensured. Let us now test the (local) *homogeneity* of $\mathcal{B}$. In this sense, by using again proposition 4.2.13, we should study the existence of a system of (local) coordinates $\left(x^i\right)$ at each $X \in \mathcal{B}_1$ such that,

$$\frac{\partial \mathcal{W}}{\partial x^l} = 0, \tag{4.24}$$

for all $l \leq 2$ if $X \neq 0$ and $l = 0$ if $X = 0$.

Let $\left(x^i\right)$ be a system of local coordinates of $\mathcal{B}_1$. Using the chain rule we have that,

$$\frac{\partial \mathcal{W}}{\partial x^i} = \frac{\partial \widehat{W}}{\partial r}\frac{\partial r}{\partial x^i} + \frac{\partial \widehat{W}}{\partial J}\frac{\partial J}{\partial x^i}.$$

Therefore, the immersion property of $\widehat{W}$ implies that $\left(x^i\right)$ are homogeneous coordinates if, and only if,

$$\textbf{(1)*} \qquad \frac{\partial r}{\partial x^l} = 0.$$

$$\textbf{(2)*} \qquad \frac{\partial J}{\partial x^l} = 0.$$

for all $l \leq 2$ if $X \neq 0$ and $l = 0$ if $X = 0$. Hence, the study of homogeneity depends only on the properties of r and J.
For each $j^1_{X,Y}\phi \in \Pi^1\left(\mathcal{B}_1, \mathcal{B}_1\right)$

$$\begin{aligned} r\left(j^1_{X,Y}\phi\right) &= g\left(Y\right)\left(T_X\phi\left(e\left(X\right)\right), T_X\phi\left(e\left(X\right)\right)\right) + \parallel X \parallel^2 \\ &= e^i\left(X\right)e^j\left(X\right)\frac{\partial \phi^k}{\partial x^i_{|X}}\frac{\partial \phi^l}{\partial x^j_{|X}}g_{kl}\left(Y\right) + \parallel X \parallel^2 \end{aligned}$$

where, in this case, e^j are the coordinates of e respect to $\left(x^i\right)$. So, considering the induced coordinates $\left(x^i, y^j, y^j_i\right)$ of $\left(x^i\right)$ on $\Pi^1\left(\mathcal{B}_1, \mathcal{B}_1\right)$ we have that for all $\left(\tilde{X}, \tilde{Y}, \tilde{F}\right)$,

$$\begin{aligned} & r \circ \left(x^i, y^j, y^j_i\right)^{-1}\left(\tilde{X}, \tilde{Y}, \tilde{F}\right) \\ = \quad & e^i\left(X\right)e^j\left(X\right)\tilde{F}^k_i\tilde{F}^l_j g_{kl}\left(Y\right) + \left(X^I\right)^2, \end{aligned}$$

where $X = \left(x^i\right)^{-1}\left(\tilde{X}\right)$ and $Y = \left(x^i\right)^{-1}\left(\tilde{Y}\right)$. In this way,

$$\frac{\partial r}{\partial x^k_{|j^1_{X,Y}\phi}} = 2\frac{\partial e^i}{\partial x^k_{|X}}e^j\left(X\right)\tilde{F}^k_i\tilde{F}^l_j g_{kl}\left(Y\right) + 2X^L\frac{\partial X^L}{\partial x^k_{|X}}.$$

So, let us study the equation,

$$\frac{\partial r}{\partial x^k} = 0.$$

Again, the dependence of the matrix variable on the left side of the equations take us to the necessary equation,

$$X^l \frac{\partial X^L}{\partial x^k_{|X}} = 0$$

Hence, by using the non-degeneracy of g we have that $\frac{\partial r}{\partial x^k} = 0$ if, and only if,

(i)

$$X^L \frac{\partial X^L}{\partial x^k} = 0 \tag{4.25}$$

(ii)

$$\frac{\partial e^i}{\partial x^k} = 0, \ \forall i. \tag{4.26}$$

Thus, **(1)*** is satisfied if, and only if,

$$\frac{\partial e^i}{\partial x^l} = 0, \qquad X^r \frac{\partial X^r}{\partial x^l} = 0, \quad \forall i, \ l \leq 2. \tag{4.27}$$

These two equations can be translated by stating that the functions e^i are constant on the spheres and the partial derivatives $\frac{\partial}{\partial x^i}$ are tangent to the spheres.
Next, let us study condition **(2)***. Notice that,

$$\begin{aligned}
& \left[J \circ \left(x^i, y^j, y^j_i\right)^{-1}\right] \left(\tilde{X}, \tilde{Y}, \tilde{F}\right) = \\
= \quad & J \left(j^1_{\left(\left(x^i\right)^{-1}\left(\tilde{X}\right), \left(y^j\right)^{-1}\left(\tilde{Y}\right)\right)} \left(\left(y^j\right)^{-1} \circ \tilde{\phi} \circ \left(x^i\right)\right)\right) \\
= \quad & J \left(\nabla_{\tilde{Y}} \left(y^j\right)^{-1}\right) J \left(\tilde{F}\right) J \left(\nabla_{\left(x^i\right)^{-1}\left(\tilde{X}\right)} \left(x^i\right)\right),
\end{aligned}$$

where $\tilde{F} = \nabla_{\tilde{X}} \tilde{\phi}$ and $\left(\tilde{X}, \tilde{Y}, \tilde{F}\right)$ is in the codomain of $\left(x^i, y^j, y^j_i\right)$. Then, $\dfrac{\partial J}{\partial x^i} = 0$ if, and only if,

$$\frac{\partial}{\partial X^I}\left(J\left(\nabla_{(x^i)^{-1}(\tilde{X})}\left(x^i\right)\right)\right) = 0. \qquad (4.28)$$

So, denoting $\left(x^i\right)^{-1}\left(\tilde{X}\right)$ by X, Eq. (4.28) is equivalent to

$$\frac{\partial^2 x^k}{\partial X^I \partial X^M_{|X}} = 0, \ \forall k, M.$$

Therefore, **(2)*** is can be expressed as follows,

$$\frac{\partial^2 x^i}{\partial X^K \partial X^M} = 0, \ \forall i, M, \ \forall K \leq 2. \qquad (4.29)$$

We conclude with this that $\mathcal{B}_1$ (locally) homogeneous if, and only if, there exists a local system of coordinates $\left(x^i\right)$ which satisfies that the functions $e^i = e\left(x^i\right)$ are constants on the spheres, the partial derivatives $\dfrac{\partial}{\partial x^i}$ are tangent to the spheres and it satisfies Eq. (4.29).
Therefore, in general, $\mathcal{B}_1$ is not (locally) homogeneous. In fact, let us consider the following vector field

$$e = X^K \frac{\partial}{\partial X^K} + r\frac{\partial}{\partial X^L}.$$

The factor $r\dfrac{\partial}{\partial X^L}$ is added to get that the vector field e does not vanish.
Assume that there exists a local system of homogeneous coordinates $\left(x^i\right)$ on $\mathcal{B}_1$. Notice that,

$$e = \left(X^K\right)\frac{\partial x^i}{\partial X^K}\frac{\partial}{\partial x^i} + r\frac{\partial x^i}{\partial X^L}\frac{\partial}{\partial x^i},$$

i.e., the coordinates e^i respect to the coordinates $\left(x^i\right)$ are given by $X^k \frac{\partial x^i}{\partial X^K} + r \frac{\partial x^i}{\partial X^L}$. Then, it should satisfy that for each $l \leq 2$

$$\begin{aligned}
\frac{\partial e^i}{\partial x^l} &= \frac{\partial}{\partial x^l}\left(X^K \frac{\partial x^i}{\partial X^K}\right) + r \frac{\partial^2 x^i}{\partial x^l \partial X^L} \\
&= \frac{\partial X^K}{\partial x^l} \frac{\partial x^i}{\partial X^k} + X^K \frac{\partial^2 x^i}{\partial x^l \partial X^K} + r \frac{\partial^2 x^i}{\partial x^l \partial X^L} \\
&= \delta_l^i + X^K \frac{\partial^2 x^i}{\partial x^l \partial X^K} + r \frac{\partial^2 x^i}{\partial x^l \partial X^L} = 0.
\end{aligned}$$

Notice that,

$$\begin{aligned}
\frac{\partial^2 x^i}{\partial x^l \partial X^L} &= \frac{\partial}{\partial X^L}\left(\frac{\partial x^i}{\partial X^L} \circ \left(x^i\right)^{-1}\right) \\
&= \frac{\partial^2 x^i}{\partial X^K \partial X^L} \frac{\partial X^K}{\partial x^l} = 0.
\end{aligned}$$

This is a consequence of Eq. (4.29). So,

$$\frac{\partial e^i}{\partial x^l} = 0,$$

if, and only if,

$$\delta_l^i = -X^K \frac{\partial^2 x^i}{\partial x^l \partial X^K}. \tag{4.30}$$

Notice that, Eq. (4.30) implies that for $i \neq l$,

$$X^k \frac{\partial^2 x^i}{\partial x^l \partial X^K} = 0.$$

Thus, considering $X = (0, 0, c)$ with $c \neq 0$ we have that

$$\begin{aligned}
\frac{\partial^2 x^i}{\partial x^l \partial X^3_{|X}} &= \frac{\partial^2 x^i}{\partial X^K \partial X^3_{|X}} \frac{\partial X^K}{\partial x^l_{|X}} \\
&= \frac{\partial^2 x^i}{\partial X^3 \partial X^3_{|X}} \frac{\partial X^3}{\partial x^l_{|X}} = 0.
\end{aligned}$$

Then $\dfrac{\partial^2 x^i}{\partial X^3 \partial X^3_{|X}} = 0$ or $\dfrac{\partial X^3}{\partial x^l_{|X}} = 0$. Observe that, by Eq. (4.30), for each $l \leq 2$

$$1 = \frac{\partial^2 x^l}{\partial X^3 \partial X^3_{|X}} \frac{\partial X^3}{\partial x^l_{|X}}.$$

So, the expression $\dfrac{\partial X^3}{\partial x^l_{|X}}$ cannot be zero. For the same reason, for $i \leq 2$, $\dfrac{\partial^2 x^i}{\partial X^3 \partial X^3_{|X}}$ is different to 0. Therefore, the above equation cannot be satisfied and the laminated simple liquid crystal $\mathcal{B}_1$ induced by this vector field e is not homogeneous.

Appendix A

Principal bundles: Frame bundles

In this appendix we do a review of the most important results and definitions related with the classical notion of *principal bundle* focusing the study on the necessary knowledge to understand the thesis. Convenient sources for a more complete exposition of principal bundles are [56]and [87]. Special attention is paid to the notions of *frame bundles* and *integrability* of its reduced subbundles. In 1950 C. Ehresmann (see [23–26]) formalized the notion of principal bundles and studied many frame bundles associated in a natural way to an arbitrary manifold: non-holonomic and holonomic frame bundles. We also remit to [5], [12], [15] and [47] for a detailed study on these topics.

Definition A.0.1. Let P be an $n-$manifold and G be a Lie group which acts over P by the right satisfying:

(i) The action of G is *free*, i.e.,

$$x \cdot g = x \Leftrightarrow g = e,$$

where $e \in G$ is the identity of G.

(ii) The canonical projection $\rho : P \to M = P/G$, where P/G is the space of orbits, is a surjective submersion.

(iii) P is *locally trivial*, i.e., P is locally a product $U \times G$, where U is an open set of M. More precisely, there exists a diffeomorphism $\Phi : \rho^{-1}(U) \to U \times G$, such that $\Phi(u) = (\rho(u), \phi(u))$, where the map $\phi : \rho^{-1}(U) \to G$ satisfies that

$$\phi(u \cdot g) = \phi(u) \cdot g, \ \forall u \in U, \ \forall g \in G.$$

Φ is called a *trivialization on* U.
A principal bundle may be analogously defined by a left action. Along the memory, we will not distinguish between a principal defined by right action and a principal bundle defined by a left action.

A principal bundle will be denoted by $P(M, G)$, or simply $\rho : P \to M$ if there is no ambiguity as to the structure group G. P is called the *total space*, M is the *base space*, G is the *structure group* and ρ is the *projection*. The closed submanifold $\rho^{-1}(x)$, $x \in M$ will be called the *fibre over* x. For each point $u \in P$, we have $\rho^{-1}(x) \triangleq uG$, where $\rho(u) = x$, and uG will be called the *fibre through* u. Every fibre is diffeomorphic to G, but this diffeomorphism depends on the choice of the trivialization.
Now, we want to define the morphism of this category.

Definition A.0.2. Given $P(M, G)$ and $P'(M', G')$ principal bundles, a principal bundle morphism from $P(M, G)$ to $P'(M', G')$ consists of a differentiable map $\Phi : P \to P'$ and a Lie group homomorphism $\varphi : G \to G'$ such that

$$\Phi(x \cdot g) = \Phi(x) \cdot \varphi(g), \ \forall x \in P, \ \forall g \in G.$$

In this case, Φ maps fibres into fibres and it induces a differentiable map $\phi : M \to M'$ by the equality $\phi(x) = \rho(\Phi(u))$, where $u \in \rho^{-1}(x)$.
If these maps are embeddings, the principal bundle morphism will be called *embedding*. In such a case, we can identify P with $\Phi(P)$, G with $\varphi(G)$ and M with $\phi(M)$ and $P(M, G)$ is said to be a *subbundle* of $P'(M', G')$. Furthermore, if $M = M'$ and $\varphi = Id_M$, $P(M, G)$ is called a *reduced subbundle* and we also say that G' *reduces* to the subgroup G.
As usual, a principal bundle morphism is called *isomorphism* if it can be

inverted by another principal bundle morphism. It is obvious that the property of being principal bundle morphism is preserved by compositions and, indeed, principal bundles next to principal bundle morphims give rise to a category denoted by $\mathcal{PB}$.

Example A.0.3. Let M be an $n-$dimensional manifold and G be a Lie group, then we can consider $M \times G$ as a principal bundle over M with projection $pr_1 : M \times G \to M$ and structure group G. The action considered here is given by,

$$(x, g)\, h = (x, gh)\,, \ \forall x \in M, \ \forall g, h \in G.$$

This principal bundle is called a *trivial principal bundle.* Equivalently, $G \times M$ may be considered as a principal bundle multiplying by the left.

Using this example we can rewrite the condition of locally trivial: In the conditions of definition A.0.1 P is locally trivial if, and only if, it is locally isomorphic (in the sense of principal bundles) to the trivial principal bundle $pr_1 : M \times G \to M$ with φ equal to the identity on G (see definition A.0.2). Note the strong resemblance to the notion of trivial Lie groupoid in example 2.2.7. This fact could gives us a clue about the close relation between transitive Lie groupoids and principal bundles.
Now, we will introduce an important example of principal bundle, the *frame bundle* of a manifold. In order to do that, we will start presenting the following definition.

Definition A.0.4. Let M be an $n-$dimensional manifold. For each point $x \in M$ an ordered basis of T_xM is called a *linear frame* at x.

Remark A.0.5. Alternatively, a linear frame at x can be viewed as a linear isomorphism $\overline{x} : \mathbb{R}^n \to T_xM$ identifying a basis on T_xM as the image of the canonic basis of $\mathbb{R}^n$ by $\overline{x}$.
$\diamond$

We may use theory of jets to give a third way to interpret a linear frame. Let us give a very brief introduction to the notion of $1-$jets of differentiable maps.
Let $\mathcal{C}^\infty(M, N)$ be the space of differentiable maps from M to N and fix

$x \in M$. Then, we may define an equivalence relation $\sim_x$ on $\mathcal{C}^\infty(M,N)$ in the following way: For each two maps $f,g \in \mathcal{C}^\infty(M,N)$,

$$f \sim_x g \;\Leftrightarrow\; f(x) = g(x) \quad \wedge \quad T_x f = T_x g,$$

where $T_x f, T_x g : T_x M \to T_{f(x)} N$ are the induced maps on tangent spaces at x of f and g respectively. The equivalence class of f respect to $\sim_x$ is called *1−jet of f at x* and is denoted $j^1_{x,y} f$ with $y = f(x)$.
The proof of the following characterization is obvious.

Lemma A.0.6. *Let us consider two differentiable maps $f,g : M \to N$ such that, for a fixed $x \in M$, $f(x) = g(x)$. Then, $j^1_{x,y} f = j^1_{x,y} g$ if, and only if, for all i,j*

$$\frac{\partial\left(y^j \circ f\right)}{\partial x^i_{|x}} = \frac{\left(\partial y^j \circ g\right)}{\partial x^i_{|x}},$$

for some local coordinates $\left(x^i\right)$ and $\left(y^j\right)$ on M and N respectivelly.

Let $j^1_{x,y} f$ and $j^1_{y,z} g$ be two 1−jets of the differentiable maps $f : M \to N$ and $g : N \to S$ at the points $x \in M$ and $y \in N$. Then, we define the *composition* $\cdot$ *of $j^1_{x,y} f$ and $j^1_{y,z} g$* as the 1−jet of the composition $g \circ f$ at x, i.e.,

$$j^1_{y,z} g \cdot j^1_{x,y} f = j^1_{x,z}(g \circ f).$$

Now, a linear frame $\overline{x} : \mathbb{R}^n \to T_x M$ at $x \in M$ (see remark A.0.5) may be considered as a 1−jet $j^1_{0,x}\phi$ at x of a local diffeomorphism ϕ from an open neighbourhood of 0 in $\mathbb{R}^n$ onto an open neighbourhood of x in M such that $\phi(0) = x$ by imposing that $T_0\phi = \overline{x}$. Here we are identifying $T_0\mathbb{R}^n$ with $\mathbb{R}^n$ via the canonical isomorphism.
Thus, we denote by FM the set of all linear frames at all the points of M. We can view FM as a principal bundle over M with structure group $Gl(n,\mathbb{R})$ and projection $\rho_M : FM \to M$ given by

$$\rho_M\left(j^1_{0,x}\phi\right) = x, \; \forall j^1_{0,x}\phi \in FM.$$

Notice that any $g \in Gl(n,\mathbb{R})$ may be canonically identified by a 1−jet $j^1_{0,0}F$ of an isomorphism F from 0 to 0. So, the right action associated to this principal bundle of $Gl(n,\mathbb{R})$ over FM is given by the composition of 1−jets.

This principal bundle is called *linear frame bundle* or simply *frame bundle of* M. Let $\left(x^i\right)$ be a local coordinate system on an open set $U \subseteq M$. Then we can introduce local coordinates $\left(x^i, x^i_j\right)$ over $FU \subseteq FM$ such that

- $x^i\left(j^1_{0,x}\phi\right) = x^i\left(x\right)$
- $y^j_i\left(j^1_{0,x}\phi\right) = \dfrac{\partial x^j \circ \phi}{\partial r^i_{|0}}$

for all $j^1_{0,x}\phi \in FU$ with $\left(r^i\right)$ the natural canonical (global) coordinates on $\mathbb{R}^n$. Notice that, by using these coordinates is straightforward to prove that ρ_M is a surjective submersion. In fact, $\left(x^i, x^i_j\right)$ induces the local trivialization from FU to $U \times Gl\left(n, \mathbb{R}\right)$. Indeed, this coordinates endows to the space $\Pi^1\left(\mathbb{R}^n, M\right)$ of all $1-$jets of all differentiable maps $f : \mathbb{R}^n \to M$ at the point 0 with $f\left(0\right) = x$ of a differentiable structure of manifold such that FM is an open subset.
Let $\psi : M \to N$ be a diffeomorphism from M to N. Then, we can defined the *first prolongation of* ψ as the isomorphism $F\psi : FM \to FN$ of principal bundles over ψ given by

$$F\psi\left(j^1_{0,x}\phi\right) = j^1_{0,y}\left(\psi \circ \phi\right), \tag{A.1}$$

for all $j^1_{0,x}\phi \in FM$ with $y = \psi\left(x\right)$. Notice that $F\psi$ is right invariant, i.e., for all $g \in Gl\left(n, \mathbb{R}\right)$ we have that

$$F\psi\left(j^1_{0,x}\phi \cdot g\right) = F\psi\left(j^1_{0,x}\phi\right) \cdot g,$$

for all $j^1_{0,x}\phi \in FN$. It is also remarkable that the inverse of $F\psi$ is given by the first prolongation of the inverse of ψ; $\left(F\psi\right)^{-1} = F\psi^{-1}$. We denote by e_1 the frame $j^1_{0,0}Id_{\mathbb{R}^n} \in F\mathbb{R}^n$, where $Id_{\mathbb{R}^n}$ is the identity map on $\mathbb{R}^n$.

Definition A.0.7. A *$G-$structure over* M, $\omega_G\left(M\right)$, is a reduced subbundle of FM with structure group G, a Lie subgroup of $Gl\left(n, \mathbb{R}\right)$.

Now, we shall introduce the notion of *integrability of a $G-$structure*. Note that there exists a principal bundle isomorphism $l : F\mathbb{R}^n \to \mathbb{R}^n \times Gl(n,\mathbb{R})$ over the identity map such that

$$l\left(j^1_{0,x}\phi\right) = \left(x, \left(J\phi_{|0}\right)\right), \ \forall j^1_{0,x}\phi \in F\mathbb{R}^n,$$

where $J\phi_{|0}$ is the Jacobian matrix of ϕ at 0. Indeed, we can consider the global section,

$$\begin{array}{cccc} s: & \mathbb{R}^n & \to & F\mathbb{R}^n \\ & x & \mapsto & j^1_{0,x}\tau_x \end{array}$$

where τ_x denote the translation on $\mathbb{R}^n$ by the vector x. So, a $1-$jet $j^1_{0,x}\phi$ can be written in a unique way as a composition of $s(x)$ and a matrix of $Gl(n,\mathbb{R})$.
We have thus obtained a principal bundle isomorphism $F\mathbb{R}^n \cong \mathbb{R}^n \times Gl(n,\mathbb{R})$ over the identity map on $\mathbb{R}^n$. Then, if G is a Lie subgroup of $Gl(n,\mathbb{R})$, we can transport $\mathbb{R}^n \times G$ by this isomorphism to obtain a $G-$structure on $\mathbb{R}^n$. These kind of $G-$structures will be called *standard flat* on $\mathbb{R}^n$.

Definition A.0.8. Let $\omega_G(M)$ be a $G-$structure over M. $\omega_G(M)$ is said to be *integrable* if we can cover M by local charts (U,φ_U) such that the restriction of the maps $l_{|F\varphi_U(U)} \circ F\varphi_U^{-1}$ to $\omega_G(M)$ are isomorphisms onto the trivial principal bundle $\varphi_U(U) \times G$ for some Lie group G.

Particularly, $\omega_G(M)$ is integrable if, and only if, for all point $x \in M$ there exists a local chart (U,φ_U) through x such that for all $j^1_{0,y}\psi \in \omega_G(M)$ with $y \in U$

$$j^1_{0,0}\left(\tau_{-\varphi_U(y)} \circ \varphi_U \circ \psi\right) \in G, \tag{A.2}$$

where $\tau_{-\varphi_U(y)}$ denote the translation on $\mathbb{R}^n$ by the vector $-\varphi_U(y)$.
Any $\{e\}-$structure on M, with e the identity of G, will be called *parallelism of M*. It is easy to show that any parallelism is, indeed, a global section of $\rho_M : FM \to M$. So, we will speak about *integrable sections*. Notice that, any parallelism $P : M \to FM$ is locally written,

$$P\left(x^i\right) = \left(x^i, P^i_j\right).$$

On the other hand, using Eq. (A.2) we have that (locally) any integrable sections can be locally written as follows

$$P_{|U} = j^1_{0,x}\left(\varphi_U^{-1} \circ \tau_{\varphi_U(x)}\right),$$

with (U, φ_U) a local chart on M. So, denoting φ_U by $\left(x^i\right)$ and using the induced coordinates $\left(x^i, x^i_j\right)$, we can write that

$$P\left(x^i\right) = \left(x^i, \delta^i_j\right).$$

So, by using right translations, we can easily prove the following result

Proposition A.0.9. *A $G-$structure $\omega_G(M)$ on M is integrable if, and only if, for each point $x \in M$ there exists a local coordinate system $\left(x^i\right)$ on M such that the local section,*

$$P\left(x^i\right) = \left(x^i, \delta^i_j\right), \tag{A.3}$$

takes values into $\omega_G(M)$.

Notice that, one could think that P can be defined globally but, however, this is not true. This result about integrability of $G-$structures can be found in [47].
Let $\Psi : F\mathbb{R}^n \to FM$ be a local isomorphism of principal bundles such that its codomain contains e_1 and the induced isomorphism on Lie groups is the identity, i.e.,

$$\Psi(\tilde{x} \cdot g) = \Psi(\tilde{x}) \cdot g, \ \forall \tilde{x} \in Dom(\Psi) \subseteq F\mathbb{R}^n,$$

for all $g \in Gl(n, \mathbb{R})$. We denote by $\psi : \mathbb{R}^n \to M$ the local diffeomorphism induced by Ψ on the base manifolds. We recall that

$$\psi \circ \rho_{\mathbb{R}^n} = \rho_M \circ \Psi.$$

The collection of all $1-$jets $j^1_{e_1,\overline{x}}\Psi$ is a manifold which will be denoted by $\overline{F}^2 M$. Of course, $j^1_{e_1,\overline{x}}\Psi$ can be canonically identified with a linear frame at the point $\overline{x}$ since $T_{e_1}\Psi : \mathbb{R}^{n+n^2} \cong T_{e_1}F\mathbb{R}^n \to T_{\overline{x}}FM$ is a linear isomorphism. Hence, we have that $\overline{F}^2 M \subset F(FM)$.
There are two canonical projections $\overline{\rho}^2_1 : \overline{F}^2 M \to FM$ and $\overline{\rho}^2 : \overline{F}^2 M \to M$ given by:

- $\overline{\rho}_1^2\left(j^1_{e_1,\overline{x}}\Psi\right)=\overline{x}$
- $\overline{\rho}^2\left(j^1_{e_1,\overline{x}}\Psi\right)=\rho_M\left(\overline{x}\right)=x$

where $\rho_M : FM \to M$ is the canonical projection of the frame bundle FM. Of course, we have $\overline{\rho}^2=\rho_M\circ\overline{\rho}_1^2$. We can show that $\overline{F}^2M$ is a principal bundle over FM with canonical projection $\overline{\rho}_1^2$ and structure group,

$$\overline{G}_1^2\left(n\right):=\{j^1_{e_1,e_1}\Psi\in\overline{F}^2\mathbb{R}^n/\ \Psi\left(e_1\right)=e_1\}={\overline{\rho}_1^2}^{-1}\left(e_1\right).$$

Notice that $\overline{G}_1^2\left(n\right)$ is a Lie subgroup of $Gl\left(n+n^2,\mathbb{R}\right)$ acting on $\overline{F}^2M$ by composition of jets.
We also have that $\overline{F}^2M$ is a principal bundle over M with canonical projection $\overline{\rho}^2$ and structure group

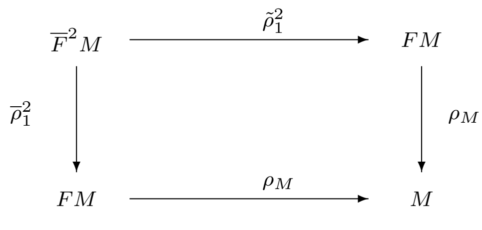
$$\overline{G}^2\left(n\right):=\{j^1_{e_1,\tilde{x}}\Psi\in\overline{F}^2\mathbb{R}^n/\ \psi\left(0\right)=0\}={\overline{\rho}^2}^{-1}\left(0\right),$$

which, again, acts on $\overline{F}^2M$ by composition of jets.
The principal bundle $\overline{F}^2M$ will be called the *non-holonomic frame bundle of second order* and its elements will be called *non-holonomic frames of second order.*

Remark A.0.10. Notice that there exists a canonical projection $\tilde{\rho}_1^2$: $\overline{F}^2M\to FM$ defined by

$$\tilde{\rho}_1^2\left(j^1_{e_1,\overline{x}}\Psi\right)=j^1_{0,x}\psi.$$

Observe that $\tilde{\rho}_1^2$ is a principal bundle morphism from $\overline{F}^2M$ to FM according to the diagram



$\diamond$

As we have done with FM, having a local coordinate system $\left(x^i\right)$ on an open set $U \subseteq M$, we can introduce local coordinates $\left(x^i, x^i_j\right)$ over $FU \subseteq FM$ and, therefore, we can introduce local coordinates $\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{,jk}, x^i_{j,k}, x^i_{j,kl}\right)$ over $F\left(FU\right)$ such that

- $x^i\left(j^1_{e_1,\overline{x}}\Psi\right) = x^i\left(\rho_M\left(\overline{x}\right)\right)$
- $x^i_j\left(j^1_{e_1,\overline{x}}\Psi\right) = x^i_j\left(\overline{x}\right)$
- $x^i_{,j}\left(j^1_{e_1,\overline{x}}\Psi\right) = \dfrac{\partial\left(x^i \circ \Psi\right)}{\partial r^j_{|e_1}}$
- $x^i_{,jk}\left(j^1_{e_1,\overline{x}}\Psi\right) = \dfrac{\partial\left(x^i \circ \Psi\right)}{\partial r^j_{k|e_1}}$
- $x^i_{j,k}\left(j^1_{e_1,\overline{x}}\Psi\right) = \dfrac{\partial\left(x^i_j \circ \Psi\right)}{\partial r^k{}_{|e_1}}$
- $x^i_{j,kl}\left(j^1_{e_1,\overline{x}}\Psi\right) = \dfrac{\partial\left(x^i_j \circ \Psi\right)}{\partial r^k_{\rho\,|e_1}}$

where $\left(r^i, r^i_j\right)$ are the induced coordinates on $F\mathbb{R}^n$ of the canonical coordinates $\left(r^i\right)$ of $\mathbb{R}^n$. Hence, if we restrict to $\overline{F}^2U$ we have that

- $x^i_{,jk} = 0$
- $x^i_{j,kl} = x^i_k \delta^j_\rho$

So, the induced coordinates on $\overline{F}^2U$ are

$$\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right). \tag{A.4}$$

Then, locally

- $\rho_M\left(x^i, x^i_j\right) = x^i$
- $\rho_{FM}\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{,jk}, x^i_{j,k}, x^i_{j,kl}\right) = \left(x^i, x^i_j\right)$
- $\overline{\rho}^2_1\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right) = \left(x^i, x^i_j\right)$
- $\overline{\rho}^2\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right) = x^i$
- $\tilde{\rho}^2_1\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right) = \left(x^i, x^i_{,j}\right)$

Notice that, like in the case of $F\mathbb{R}^n$, there exists a canonical isomorphism $\bar{l} : \overline{F}^2\mathbb{R}^n \to \mathbb{R}^n \times \overline{G}^2(n)$. In fact, let us define a global section $\overline{s} : \mathbb{R}^n \to \overline{F}^2\mathbb{R}^n$ as follows,

$$\overline{s}(x) = j^1_{e_1, e_{1x}} F\tau_x,$$

where τ_x denote the translation on $\mathbb{R}^n$ by the vector x. Then, the expression of $\overline{s}$ in coordinates is,

$$\overline{s}\left(x^i\right) = \left(\left(x^i, \delta^i_j\right), \delta^i_j, 0\right).$$

So, a non-holonomic frame of second order $j^1_{e_1,\overline{x}}\Psi$ at a point $x \in \mathbb{R}^n$ may be written in a unique way as

$$j^1_{e_1,\overline{x}}\Psi = \overline{s}(x) \cdot \overline{g},$$

where $\overline{g} \in \overline{G}^2(n)$ such that $\bar{l}\left(j^1_{e_1,\overline{x}}\Psi\right) = (x, \overline{g})$. We have thus obtained the principal bundle isomorphism $\bar{l} : \overline{F}^2\mathbb{R}^n \cong \mathbb{R}^n \times \overline{G}^2(n)$. Now, if $\overline{G}$ is a Lie subgroup of $\overline{G}^2(n)$, we can transport $\mathbb{R}^n \times \overline{G}$ by this isomorphism to obtain a $\overline{G}-$reduction of $\overline{F}^2\mathbb{R}^n$.

Definition A.0.11. Let $\overline{G}$ be a Lie subgroup of $\overline{G}^2(n)$. A *second-order non-holonomic $\overline{G}-$structure* $\overline{\omega}_{\overline{G}}(M)$ is a reduced subbundle of $\overline{F}^2M$ with structure group $\overline{G}$.

Hence, the $\overline{G}-$reduction of $\overline{F}^2\mathbb{R}^n$ obtained above is a second-order non-holonomic $\overline{G}-$structure on $\mathbb{R}^n$ which will be called the *standard flat second-order non-holonomic* $\overline{G}-$*structure.*

Next, we will introduce the notion of *integrability of a second-order non-holonomic* $\overline{G}-$*structure.*

Definition A.0.12. Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}-$structure on M. $\overline{\omega}_{\overline{G}}(M)$ is said to be *integrable* if it is locally isomorphic to the trivial principal bundle $\overline{G}\times\mathbb{R}^n$ via local charts on M, or equivalently, it is locally isomorphic to the standard flat $\overline{G}-$structure on $\mathbb{R}^n$.

What we mean by "locally isomorphic" is that for each point $x \in M$, there exists a local chart through x, $\varphi_U : U \to \overline{U}$ such that induces an isomorphism of principal bundles given by

$$\Psi_U : \overline{\omega}_{\overline{G}}(U) \to \overline{U} \times \overline{G},$$

where

$$\Psi_U\left(j^1_{e_1,\overline{x}}\Psi\right) = \Big(\varphi_U(x), j^1_{e_1,\tilde{x}_0}\left(F\left(\tau_{-\varphi_U(x)} \circ \varphi_U\right)\right) \circ \Psi\Big),$$

with $\rho_M(\overline{x}) = x$. Notice that, analogously to the case of integrable $G-$structures, we can express the isomorphism Ψ_U as follows

$$\Psi_U = \bar{l} \circ F(F\varphi_U).$$

Remark A.0.13. There exists an alternative definition of second-order non-holonomic frames (see [77]). Consider a differentiable map $\phi : U \to FM$ defined on some open neighbourhood of 0 in $\mathbb{R}^n$ such that $\rho_M \circ \phi$ is an embedding. Then the $1-$jet $j^1_{0,\phi(0)}\phi$ is a non-holonomic frame of second order at $x = \rho_M(\phi(0))$. In fact, given ϕ we define a local principal bundle isomorphism $\Phi : F\mathbb{R}^n \to FM$ over U given by

$$\Phi(r, R) = \phi(r)R,$$

where $r \in \mathbb{R}^n$ and $R \in Gl(n, \mathbb{R})$. Thus, $j^1_{e_1,\overline{x}}\Phi$ defines a non-holonomic frame of second order at x. Conversely, having a local principal bundle isomorphism $\Phi : F\mathbb{R}^n \to FM$ over an open set U, we define ϕ as follows:

$$\phi(r) = \Phi(r, e),$$

where $r \in \mathbb{R}^n$ and $e \in Gl(n, \mathbb{R})$ is the identity.

$\diamond$

Any second-order non-holonomic $\{\overline{e}\}$−structure on M, with $\overline{e}$ the identity of $\overline{G}^2(n)$, will be called *non-holonomic parallelism of second-order*. It is easy to show that any non-holonomic parallelism of second-order is, indeed, a global section of the second-order non-holonomic frame bundle $\overline{F}^2M$. So, we can consider *integrable sections*. Let (x^i) be a local coordinate system on an open set $U \subseteq M$ and $\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right)$ be the induced coordinates on $\overline{F}^2U$. So, a non-holonomic parallelism of second-order $\overline{P}$ is written locally

$$\overline{P}(x^i) = \left(\left(x^i, P^i_j\right), P^i_{,j}, R^i_{j,k}\right).$$

Therefore, we have that (locally) any integrable sections can be written as follows

$$\overline{P}(x^i) = \left(\left(x^i, \delta^i_j\right), \delta^i_j, 0\right).$$

respect to a particular local coordinates (x^i). Indeed, we can show that

Proposition A.0.14. *Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}$−structure on M. $\overline{\omega}_{\overline{G}}(M)$ is integrable if, and only if, for each point $x \in M$ there exists a local coordinate system (x^i) on M such that the local section,*

$$\overline{P}(x^i) = \left(\left(x^i, \delta^i_j\right), \delta^i_j, 0\right), \tag{A.5}$$

takes values into $\overline{\omega}_{\overline{G}}(M)$.

Notice that, in a similar way to the case of the integrable G−structures in the frame bundle, Eq. (A.5) is equivalent to the following: for each $z \in M$, there exists a local chart (φ_U, U) over z such that for all $x \in U$

$$\overline{P}(x) = j^1_{e_1,\overline{x}}\left(F\left(\varphi_U^{-1} \circ \tau_{\varphi_U(x)}\right)\right), \tag{A.6}$$

where $\tau_{\varphi_U(x)}$ denotes the translation on $\mathbb{R}^n$ by the vector $\varphi_U(x)$.

Next, we shall describe a particular subbundle of $\overline{F}^2 M$. Consider the non-holonomic frames of second order given by $j^1_{e_1,\overline{x}}(F\psi)$, where $\psi : \mathbb{R}^n \to M$ is a local diffeomorphism from 0 to $\rho_M(\overline{x}) = x \in M$. These kind of frames are called *holonomic frames of second order*. The set of all holonomic frames of second order is denoted by F^2M and it is called *second-order holonomic frame bundle*. The restrictions of $\overline{\rho}^2_1$ and $\overline{\rho}^2$ to F^2M are denoted by $\rho^2_1 : F^2M \to FM$ and $\rho^2 : F^2M \to M$. ρ^2_1 endows to F^2M with a principal bundle structure with structure group $G^2_1(n)$, which is the set of all $1-$jets of local isomorphism of the form $F\psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeomorphism with $F\psi(e_1) = e_1$ (equivalently, $j^1_{0,0}\phi = e_1$).
$\rho^2 : F^2M \to M$ is also a principal bundle which structure group $G^2 = (\rho^2)^{-1}(0)$.
We deduce that ρ^2_1 (resp. ρ^2) is a principal subbundle of $\overline{\rho}^2_1$ (resp. $\overline{\rho}^2$). So, restricting the isomorphism $\overline{l} : \overline{F}^2\mathbb{R}^n \cong \mathbb{R}^n \times \overline{G}^2(n)$ we have that

$$\tilde{l} : F^2\mathbb{R}^n \cong \mathbb{R}^n \times G^2(n).$$

Then, for each Lie subgroup G of $G^2(n)$ we obtain a $G-$reduction of $F^2\mathbb{R}^n$ which is isomorphic to $\mathbb{R}^n \times G$.

Definition A.0.15. Let G be a Lie subgroup of $G^2(n)$. A *second-order holonomic $G-$structure $\omega_G(M)$* is a reduced subbundle of F^2M with structure group G.

Hence, the $G-$reduction of $F^2\mathbb{R}^n$ obtained above is a second-order holonomic $G-$structure on $\mathbb{R}^n$ which will be called the *standard flat second-order holonomic $G-$structure.*

Note that each second-order holonomic $G-$structure $\omega_G(M)$ can be seen as a second-order non-holonomic $G-$structure. So, the notion of integrability will be the same.
A *holonomic parallelism of second order* is a second-order holonomic trivial structure or, equivalently, a global section of $\rho^2 : F^2M \to M$. So,

we will also speak about *integrable sections of* F^2M.

Observe that, by definition, any integrable non-holonomic parallelism of second order is in fact holonomic.
Summarizing we have the following sequence of Lie subgroups:

$$G^2(n) \subset \overline{G}^2(n) \subset Gl(n,\mathbb{R}) \times Gl\left(n+n^2,\mathbb{R}\right),$$

$$G_1^2(n) \subset \overline{G}_1^2(n) \subset Gl\left(n+n^2,\mathbb{R}\right),$$

and the following sequence of principal bundles:

$$F^2M \subset \overline{F}^2M \subset F(FM),$$

over FM and

$$F^2M \subset \overline{F}^2M,$$

over M.
Let $(x)^i$ be a local coordinate system on an open $U \subseteq M$. Then, we can induce local coordinates over $\overline{F}^2M$ $\left(\left(x^i, x^i_j\right), x^i_{,j}, x^i_{j,k}\right)$ (see Eq. (A.4)). Then, if we restrict to F^2M we have that

$$x^i_j = x^i_{,j} \qquad ; \qquad x^i_{j,k} = x^i_{k,j}.$$

Thus, we may obtain local coordinates on F^2M denoted as follows:

$$\left(x^i, x^i_j, x^i_{jk}\right), \quad x^i_{jk} = x^i_{j,k} = x^i_{k,j}. \tag{A.7}$$

Let us now define the so-called *non-holonomic prolongations of parallelisms of second order*. To do this, we shall describe a method to prolongate a pair of ordinary parallelisms in order to obtain a non-holonomic parallelism of second order.
Let M be a manifold and $\overline{P}$ be a section of the second-order non-holonomic frame bundle $\overline{F}^2M$. Then, $\overline{P}\left(x^i\right) = \left(\left(x^i, P^i_j\right), P^i_{,j}, R^i_{j,k}\right)$ induces two sections P and Q of FM (i.e. induces two ordinary parallelisms on M) by projecting $\overline{P}$ via the two canonical projections $\overline{\rho}^2_1$ and $\tilde{\rho}^2_1$, i.e.,

$$P = \overline{\rho}^2_1 \circ \overline{P}, \qquad Q = \tilde{\rho}^2_1 \circ \overline{P}.$$

So, we obtain

$$P\left(x^i\right) = \left(x^i, P^i_j\right), \qquad Q\left(x^i\right) = \left(x^i, P^i_{,j}\right).$$

Conversely, let $P, Q : M \to FM$ be two sections of FM. Hence, P (resp. Q) defines a family of n (where n is the dimension of M) linearly independent vector fields $\{P_1, \cdots, P_n\}$ (resp. $\{Q_1, \cdots, Q_n\}$).
We define a horizontal subspace $H_{P(x)}$ at the point $P(x)$ by translating the basis $\{Q_a(x)\}$ at x into a set of linearly independent tangent vectors at $P(x)$,

$$\{T_x P\left(Q_a\left(x\right)\right)\}.$$

Locally,

$$T_x P\left(Q^i_a\left(x\right)\frac{\partial}{\partial x^i_{|x}}\right) = Q^i_a\left(x\right)\frac{\partial}{\partial x^i_{|P(x)}} + Q^i_a\left(x\right)\frac{\partial P^r_s}{\partial x^i_{|x}}\frac{\partial}{\partial x^r_{s|P(x)}},$$

where,

$$P_a\left(x\right) = P^i_a\left(x\right)\frac{\partial}{\partial x^i_{|x}}, \qquad Q_a\left(x\right) = Q^i_a\left(x\right)\frac{\partial}{\partial x^i_{|x}}.$$

By completing this set of linearly independent tangent vectors to a basis of $T_{P(x)}FM$ we obtain a second-order non-holonomic frame at x. We have so obtained a section of $\overline{F}^2M$ (i.e. a non-holonomic parallelism of second order on M), which is denoted by $P^1(Q)$.

Definition A.0.16. A non-holonomic parallelism of second order $\overline{P}$ is said to be a *prolongation* if $\overline{P} = P^1(Q)$ where P and Q are the induced ordinary parallelisms.

The local expression of $P^1(Q)$ becomes

$$P^1\left(Q\right)\left(x^i\right) = \left(\left(x^i, P^i_j\right), Q^i_j, Q^l_k\frac{\partial P^i_j}{\partial x^l}\right). \tag{A.8}$$

Hence, a section of $\overline{F}^2M$, $\overline{P}\left(x^i\right) = \left(\left(x^i, P^i_j\right), Q^i_j, R^i_{j,k}\right)$, is a second-order non-holonomic prolongation if and only if

$$R^i_{j,k} = Q^l_k\frac{\partial P^i_j}{\partial x^l}. \tag{A.9}$$

Remark A.0.17. Now, we will describe another way to construct $P^1(Q)$ which is going to be useful. Let $P, Q : M \to FM$ be two sections and we denote

$$Q(x) = j^1_{0,x}\psi_x.$$

Then, for each $a = 1, \cdots, n$

$$Q_a(x) = T_0\psi_x\left(\frac{\partial}{\partial r^a_{|0}}\right),$$

which implies that

$$T_xP(Q_a(x)) = T_0(P\circ\psi_x)\left(\frac{\partial}{\partial r^a_{|0}}\right).$$

Taking into account this equality we construct the following map

$$\begin{array}{rccl} \overline{P\circ\psi_x}: & FU & \to & FV \\ & j^1_{0,v}f & \mapsto & P(\psi_x(v))\cdot j^1_{0,0}(\tau_{-v}\circ f). \end{array}$$

where $\psi_x : U \to V$. It easy to show that $\overline{P\circ\psi_x}$ is an isomorphism of principal bundle over ψ_x with inverse given by

$$j^1_{0,w}g \in FV \mapsto j^1_{0,\psi_x^{-1}(w)}\tau_{\psi_x^{-1}(w)}\cdot[P(w)]^{-1}\cdot j^1_{0,w}g.$$

Thus, we can define,

$$\begin{array}{rccl} \overline{P}: & M & \to & \overline{F}^2M \\ & x & \mapsto & j^1_{e_1,\overline{x}}\left(\overline{P\circ\psi_x}\right) \end{array}$$

which satisfies

(i) $\overline{\rho}^2_1\circ\overline{P}(x) = \overline{P\circ\psi_x}(e_1) = P(x)$

(ii) $\tilde{\rho}^2_1\circ\overline{P}(x) = j^1_{0,x}\psi_x = Q(x)$

(iii) $\dfrac{\partial\left(P_j^i \circ \psi_x\right)}{\partial x^k_{|0}} = dP^i_{j\,|x} \circ \dfrac{\partial \psi_x}{\partial x^k_{|0}} = dP^i_{j\,|x} \circ \left(Q_k^1(x), \cdots, Q_k^n(x)\right)$

Then, by definition of induced coordinates, $R^i_{j,k}$ is given by

$$R^i_{j,k}(x) = Q^l_k(x)\,\frac{\partial P^i_j}{\partial x^l_{|x}}.$$

Therefore $\overline{P} = P^1(Q)$.

Notice that, if Q is integrable, then there exists local coordinates $\left(x^i\right)$ on M such that

$$Q^i_j = \delta^i_j,$$

and hence,

$$R^i_{j,k} = \frac{\partial P^i_j}{\partial x^k},$$

where $P^1(Q) = \left(x^i, P^i_j, Q^i_j, R^i_{j,k}\right)$. In such a case, $P^1(Q)$ is said to be *integrable*. However, in general $P^1(Q)$ is not integrable as a paralellism (see proposition A.0.18).

In this case for each $z \in M$, there exists a local chart (ψ_U, U) over z such that for all $x \in U$

$$\begin{aligned} P^1(Q)(x) &= j^1_{e_1,\overline{x}}\left(\overline{P \circ \left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)}\right) \\ &= j^1_{e_1,\overline{x}}\left(\overline{P \circ \psi_U^{-1}} \circ F\tau_{\psi_U(x)}\right) \end{aligned}$$

In fact, for each $j^1_{0,v} f \in F\psi_U(U)$ we have

$$\begin{aligned} &\overline{P \circ \left(\psi_U^{-1} \circ \tau_{\psi_U(x)}\right)}\left(j^1_{0,v} f\right) = \\ &= P\left(\psi_U^{-1}\left(v + \psi_U(x)\right)\right) \cdot j^1_{0,0}\left(\tau_{-v} \circ f\right) \\ &= \left(\overline{P \circ \psi_U^{-1}} \circ F\tau_{\psi_U(x)}\right)\left(j^1_{0,v} f\right) \end{aligned}$$

Thus, let $P^1(Q)$ be a non-holonomic integrable prolongation of second-order; then, for each $z \in M$, there exists a local principal bundle isomorphism Ψ from an open set $FU \subseteq FM$ with $z \in U$ to an open subset of $F\mathbb{R}^n$ such that for all $x \in U$

$$P^1(Q)(x) = j^1_{e_1,\overline{x}}\left(\Psi^{-1} \circ F\tau_{\psi(x)}\right), \tag{A.10}$$

where ψ is the induced map of Ψ onto the base manifolds. On the other hand, using ψ as a local chart we can prove that Eq. (A.10) implies that $P^1(Q)$ is a non-holonomic integrable prolongation of second order.
This equality reminds us Eq. (A.6), for second-order non-holonomic integrable sections. Indeed, *a second-order non-holonomic integrable prolongation $P^1(Q)$ satisfying Eq. (A.10) is integrable if and only if we have*

$$j^1_{\overline{x},e_{1\,\psi(x)}}\Psi = j^1_{\overline{x},e_{1\,\psi(x)}}F\psi,$$

for all x, where $e_{1\,\psi(x)} = j^1_{0,\psi(x)}\tau_{\psi(x)}$. Thus, a second-order non-holonomic prolongation is integrable if and only if takes values in the holonomic frame, i.e., the only integrable prolongations in F^2M are the integrable sections.

In general, we can think about the second-order non-holonomic integrable prolongations as an intermediate step beetwen sections and integrable sections. The following result is obvious.

Proposition A.0.18. *Let $\overline{P}$ be a section of $\overline{F}^2M$. $\overline{P}$ is integrable if and only if $\overline{P} = P^1(Q)$ is a second-order non-holonomic integrable prolongation and $P = Q$. In particular, a second-order non-holonomic integrable prolongation $P^1(Q)$ is integrable if and only if $P = Q$.*

This result provides us examples of second-order no-holonomic integrable prolongations which are not integrable. Indeed, any prolongation of two different ordinary parallelisms is not a second-order no-holonomic integrable prolongation.

Now, to end this appendix, we will define the concept prolongation for non-holonomic $\overline{G}-$structures of second order.

Definition A.0.19. Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}-$structure. $\overline{\omega}_{\overline{G}}(M)$ is a non-holonomic integrable prolongation of second-order if we can cover M by local non-holonomic integrable prolongations of second order which take values in $\overline{\omega}_{\overline{G}}(M)$.

Notice that Definition A.0.19 can be expressed as follows: For any point $x \in M$ there exists a local coordinate system (x^i) over an open set $U \subseteq M$ which contains x such that the local section on U

$$P^1(Q)(x^i) = \left(x^i, P^i_j, \delta^i_j, \frac{\partial P^i_j}{\partial x^k}\right),$$

is contained in $\overline{\omega}_{\overline{G}}(M)$.

As we have noticed, a second-order non-holonomic $\overline{G}-$structure which is contained in F^2M is integrable if and only if it is an integrable prolongation.

Remark A.0.20. Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}-$structure. Then, we could define a non-holonomic integrable prolongation of second order in a similar way to integrable $\overline{G}-$structures. In fact, using Eq. (A.10), we can prove that $\overline{\omega}_{\overline{G}}(M)$ *is a non-holonomic integrable prolongation of second-order if, and only if, for all point* $x \in M$, *there exists a local isomorphism of principal bundles whose isomorphism of Lie groups is the identity map,* $\Phi : FU \to F\overline{U}$, *with* $x \in U$ *such that it induces an isomorphism of principal bundles given by*

$$\Upsilon : \overline{\omega}_{\overline{G}}(U) \to \overline{U} \times \overline{G},$$

where $\Upsilon\left(j^1_{e_1,\overline{z}}\Phi\right) = \left(\psi(z), \overline{\Upsilon}\left(j^1_{e_1,\overline{z}}\Phi\right)\right)$ *and*

$$\overline{\Upsilon}\left(j^1_{e_1,\overline{z}}\Phi\right) = j^1_{e_1,\overline{z}_0}\left(F\left(\tau_{-\psi(z)}\right) \circ \Psi \circ \Phi\right),$$

with ψ *the induced map of* Ψ *over the base manifold and* $\rho_M(\overline{z}) = z$. Thus, $\overline{\omega}_{\overline{G}}(M)$ is a non-holonomic integrable prolongation of second-order if it is locally isomorphic to the trivial principal bundle $\mathbb{R}^n \times \overline{G}$ by a more general class of local charts (see definition A.0.12).

Proposition A.0.21. *Let $\overline{\omega}_{\overline{G}}(M)$ be a second-order non-holonomic $\overline{G}$−structure. If $\overline{\omega}_{\overline{G}}(M)$ is integrable, then $\overline{\omega}_{\overline{G}}(M)$ is a non-holonomic integrable prolongation of second-order.*

Not all non-holonomic integrable prolongation of second-order is integrable (see proposition A.0.18).

It directly follows that if $\overline{\omega}_{\overline{G}}(M)$ is a second-order non-holonomic integrable prolongation, then the projected G−structure by $\tilde{\rho}_1^2$ is integrable.

Appendix B

Foliations and distributions

This part of the appendix is devoted to give a clear introduction *foliations* and *distributions*. All the results and definitions exposed here can be found in [71] and [20] (see also [84]).
Intuitively speaking, a foliation of a manifold M is a decomposition of M into immersed submanifolds, the leaves of the foliation, which, in some way explained below, fits together nicely. These leaves are not necessarily of the same dimension.

Definition B.0.1. A *smooth singular foliation* or simply *foliation* on a smooth manifold M is a partition $\mathcal{F} := \{\mathcal{F}(x)\}$ of M into a disjoint union of smooth immersed connected submanifolds $\mathcal{F}(x)$, called *leaves*, which satisfies the following local foliation property at each point $x \in M$: Denote the leaf that contains x by $\mathcal{F}(x)$, the dimension of $\mathcal{F}(x)$ by k and the dimension of M by n.
Then there is a smooth local chart of M with coordinates $\left(y^1, \ldots, y^n\right)$ in a neighborhood U of x, $U := \{-\epsilon < y^1 < \epsilon, \ldots, -\epsilon < y^n < \epsilon\}$, such that the $k-$dimensional disk $\{y^{k+1} = \ldots = y^n = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}(x)$ with U which contains x, and each $k-$dimensional disk $\{y^{k+1} = c_{k+1}, \ldots y^n = c_n\}$,

where $c_{k+1}, \ldots, c_n$ are constants, is wholly contained in some leaf of $\mathcal{F}$. This (local) chart is called the *foliation chart* of $\mathcal{F}$ at x. A *foliation atlas* of dimension k is an atlas of M given by foliation charts. The *dimension* of the foliation $\mathcal{F}$ is a map $dim : M \to \{0, 1, \ldots, n\}$ such that, for each $x \in M$, $dim\,(x)$ is the dimension of the leaf $\mathcal{F}\,(x)$ at x.
If all the leaves of a singular foliation $\mathcal{F}$ have the same dimension, then one says that $\mathcal{F}$ is a *regular foliation*. Furthermore, in this case de dimension of $\mathcal{F}$ is a constant map and, hence, is identified with a number k equal to dimension of the leaves of $\mathcal{F}$.

Let M be a manifold and $\mathcal{F}$ be a foliation on M. Consider a foliation chart of M at a point $x \in M$ with coordinates $\left(y^1, \ldots, y^n\right)$ in a neighborhood $U := \{-\epsilon < y^1 < \epsilon, \ldots, -\epsilon < y^n < \epsilon\}$ of x. Then, the subset $U_0 := \{y^{k+1} = \ldots = y^n = 0\}$ of U given by the condition of that the last $n-k$ coordinates are 0 is an open subset of the leaf $\mathcal{F}\,(x)$ (with the induced topology) which contains to x. Indeed, we may define a local chart of $\mathcal{F}\,(x)$ at x over U_0 by restricting the map $\left(y^1, \ldots, y^k\right)$, where k is the dimension of $\mathcal{F}\,(x)$, to U_0. If $\mathcal{F}$ is a regular foliation, the restriction of the map $\left(y^1, \ldots, y^k\right)$ to each of the subsets $U_{c_1,\ldots,c_k} := \{y^{k+1} = c_{k+1}, \ldots y^n = c_n\}$, with c_i constant for all $i = k+1, \ldots n$, defines a local chart of the leaf in which $U_{c_1,\ldots,c_k} := \{y^{k+1} = c_{k+1}, \ldots y^n = c_n\}$ is contained.

Definition B.0.2. Let M be a smooth manifold and $\mathcal{F}$ be a foliation of M. The *space of leaves* $M/\mathcal{F}$ is the quotient space of M, obtained by identifying two points of M if they lie on the same leaf of $\mathcal{F}$.

Now, we will define a category associated to the foliations of the smooth manifolds. Thus, we need define the objects and the morphisms.

Definition B.0.3. A *foliated manifold* is a pair $(M, \mathcal{F})$, where M is a smooth manifold and $\mathcal{F}$ a foliation of M. If the foliation $\mathcal{F}$ is regular, the pair $(M, \mathcal{F})$ will be called *regular foliated manifold*.
A *morphism between foliated manifolds* $f : (M, \mathcal{F}) \to (N, \mathcal{S})$ is a smooth map $f : M \to N$ which maps leaves of $\mathcal{F}$ into the leaves of $\mathcal{S}$.

Thus, since it is obvious that the composition of morphisms between foliated manifolds is again a morphism between foliated manifolds, we may construct a category of foliated manifolds. We will denote this category by $\mathcal{FM}$.

Note that, in the particular case of a regular foliated manifold $(M, \mathcal{F})$, the foliation $\mathcal{F}$ determines a foliation atlas such that the change-of-charts diffeomorphisms of foliation charts φ_i, φ_j are of the form

$$\varphi_i \circ \varphi_j^{-1}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \tag{B.1}$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.
Conversely, Let $\{(U_i, \varphi_i)\}$ be an atlas of a manifold M such that the change-of-charts diffeomorphisms of φ_i, φ_j satisfy Eq. (B.1). Then, each U_i is divided into *plaques*, which are the connected components of the submanifolds $\varphi_i^{-1}(\mathbb{R}^k \times \{y\})$, $y \in \mathbb{R}^{n-k}$, and the change-of-chart diffeomorphisms preserve this division. The plaques globally amalgamate into leaves. Thus, $\{(U_i, \varphi_i)\}$ determines a unique regular foliation.
So, we have proved the following result:

Proposition B.0.4. *Let M be a manifold with an atlas $\{(U_i, \varphi_i)\}_i$. $\{(U_i, \varphi_i)\}_i$ is a foliation atlas associated to a unique regular foliation $\mathcal{F}$ of M if, and only if, the changes of coordinates satisfy Eq. (B.1).*

In this way, we can give another equivalent definition:

Proposition B.0.5. *A regular foliation $\mathcal{F}$ of a manifold M can be equivalently described in the following way: An open cover $\{U_i\}$ of M with submersions $s_i : U_i \to \mathbb{R}^{n-k}$ (where k is the dimension of $\mathcal{F}$) such that there are diffeomorphisms (necessarily unique)*

$$\gamma_{ij} : s_j(U_i \cap U_j) \to s_i(U_i \cap U_j),$$

with $\gamma_{ij} \circ s_{j|U_i \cap U_j} = s_{i|U_i \cap U_j}$.
Note that, by unicity, the diffeomorphisms γ_{ij} satisfy the cocycle condition $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$.

Proof. If (U_i, s_i, γ_{ij}) is a triple on M satisfying condition of above, using the rank theorem, we can construct an atlas $\{(V_j, \varphi_j)\}_j$ so that each V_j is a subset of an U_{α_j} and there exists a diffeomorphism $\psi_j : s_{\alpha_j}(V_j) \to \mathbb{R}^{n-k}$, such that

$$\psi_j \circ s_{\alpha_j} = pr_2 \circ \varphi_j,$$

where $pr_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the projection on the last $(n-k)$ components. This atlas is a foliation atlas. In fact, if $(x,y) \in \varphi_i(U_i \cap U_j) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$

$$\begin{aligned} pr_2 \circ \varphi_j \circ \varphi_i^{-1}(x,y) &= \left(\psi_j \circ s_{\alpha_j} \circ \varphi_i^{-1}\right)(x,y) \\ &= \left(\psi_j \circ \gamma_{\alpha_j \alpha_i} \circ s_{\alpha_i} \circ \varphi_i^{-1}\right)(x,y) \\ &= \left(\psi_j \circ \gamma_{\alpha_j \alpha_i} \circ \psi_i^{-1}\right)(y). \end{aligned}$$

Conversely, if $\{(U_j, \varphi_j)\}$ is an atlas such that the change-of-chart diffeomorphisms are of the form of Eq. (B.1), we take $s_j = pr_2 \circ \varphi_j$ and $\gamma_{ij} = h_{ij}$ (see Eq. (B.1)). $\square$

A triple (U_i, s_i, γ_{ij}) satisfying conditions of proposition B.0.5 is called *the Haefliger cocycle representing* $\mathcal{F}$.

Example B.0.6. The space $\mathbb{R}^n$ admits a (regular) foliation of dimension k, for which the foliation atlas consists of only one chart, the identity map $Id : \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then, the leaves are

$$\mathbb{S}(x_0, y_0) = \{(x, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in \mathbb{R}^k\},$$

for each $(x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Generally, consider a map $f : \mathbb{R}^n \to \{0, 1, \ldots, n\}$ such that if $f\left(x^1, \ldots, x^n\right) = k$ it satisfies that

$$f\left(y^1, \ldots, y^k, x^{k+1}, \ldots, x^n\right) = k.$$

Then, there exists a unique foliation $\mathbb{S}$ of $\mathbb{R}^n$ with f as the dimension map. In fact, the leaves of this foliation are given by

$$\mathbb{S}\left(x^i, \ldots, x^n\right) := \{\left(y^i, x^j\right) \ : \ y^i \in \mathbb{R}, \ i = 1, \ldots, k\},$$

where $f\left(x^1, \ldots, x^n\right) = k$. This foliation will be called *trivial o canonical foliation of* $\mathbb{R}^n$ *associated to* f.

Now, we will give a result which justify the name of *canonical foliation.*

Proposition B.0.7. *Let M be a manifold and $\mathcal{F}$ be a partition into subsets of M. Denoting by $\mathcal{F}(x)$ the subset of the partition $\mathcal{F}$ which contains to $x \in M$, we have that $\mathcal{F}$ is a foliation if, and only if, there exists a canonical foliation $\mathbb{S}$ of $\mathbb{R}^n$ such that we can cover M by local charts of $\varphi : U \to \mathbb{R}^n$ satisfying that*

$$\mathbb{S}(\varphi(x)) \cap \varphi(U) = \varphi(\mathcal{F}(x) \cap U),$$

for all $y \in U$.

Proof. The proof follows from the definition. □

In a more categorical way we can summarize the above result as follows: *a partition $\mathcal{F}$ of a manifold M is a foliation if, and only if, $(M, \mathcal{F})$ is locally isomorphic (in the categorical sense) to some canonical foliated manifold $(\mathbb{R}^n, \mathbb{S})$.*

Example B.0.8. Any submersion $\pi : M \to N$ defines a regular foliation of M whose leaves are the connected components of the fibres of π. The dimension of the leaves is equal to the codimension of N. An foliation atlas is derived of the rank theorem. Foliations associated to the submersions are also called *simple foliations*. The foliations associated to submersions with connected fibres are called *strictly simple*.
Note that, it is easy to prove that a simple foliation is strictly simple precisely when its space of leaves is Hausdorff.

Example B.0.9. Let $(M, \mathcal{F})$ and $(N, \mathcal{S})$ be two foliated manifolds. Then there is the *product foliation* $\mathcal{F} \times \mathcal{S}$ on $M \times N$, which leaves are the products of leaves of $\mathcal{F}$ and $\mathcal{S}$. Furthermore,

$$T(\mathcal{F} \times \mathcal{S}) = T\mathcal{F} \oplus T\mathcal{S}.$$

Example B.0.10. Let $f : N \to M$ be a smooth map and $\mathcal{F}$ a regular foliation of M of dimension k. Assume that f is transversal to $\mathcal{F}$. Then we can get a foliation $f^{-1}(\mathcal{F})$ on N as follows.
Suppose that $\mathcal{F}$ is given by the Haefliger cocycle (U_i, s_i, γ_{ij}) on M. Put $V_i = f^{-1}(U_i)$ and $s_i' = s_i \circ f_{|V_i}$. Note that, for each $x \in N$ such that $f(x) \in U_i$

$$T_x s_i' = T_{f(x)} s_i \circ T_x f,$$

is surjective. In fact, $T_{f(x)}s_i$ is surjective and trivial on $T_{f(x)}\mathcal{F}$ (see proof of the proposition (B.0.5)). Thus, using that

$$T_x f\left(T_x N\right) + T_{f(x)}\mathcal{F} = T_{f(x)}M,$$

it is easy to prove that $T_x s_i'$ is surjective.
The foliation $f^{-1}\left(\mathcal{F}\right)$ is now given by the Haefliger cocycle $\left(V_i, s_i', \gamma_{ij}\right)$ on N. We have that

$$codim\left(f^{-1}\left(F\right)\right) = codim\left(\mathcal{F}\right),$$

and,

$$T\left(f^{-1}\left(\mathcal{F}\right)\right) = \left(Tf\right)^{-1}\left(T\mathcal{F}\right).$$

As a consequence, N is foliated by connected components of $f^{-1}\left(\mathcal{F}\left(x\right)\right)$, where $\mathcal{F}\left(x\right)$ are the leaves of $\mathcal{F}$.

Example B.0.11. Let $\phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. We say that the action of G on M is *foliated* if $dim\left(G_x\right)$ (where G_x is the isotropy group on x) is a constant function of x. In this case the connected components of the orbits of the action are the leaves of a regular foliation of M.

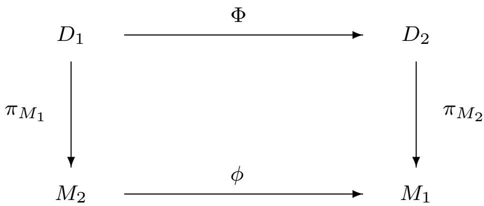
Let $\left(M, \mathcal{F}\right)$ be a foliated manifold. Then, the tangent spaces of the leaves of the foliation defines a map which associates to any point x of M a subspace of the tangent space $T_x M$. So, we obtain what is called a *distribution on M* (see example B.0.14).

Definition B.0.12. A *singular distribution* or simply *distribution D* on a smooth manifold M is the assignment to each point x of M a vector subspace D_x, called *fibre* at x, of the tangent space $T_x M$. If the dimension of D_x is constant the distribution is called a *regular distribution.*
The distribution D is called *smooth* if for any point x of M and any vector $v_x \in D_x$, there is a smooth vector field Θ defined in a neighborhood U_x of x which is tangent to the distribution, i.e.,

$$\Theta\left(y\right) \in D_y, \ \forall y \in U_x,$$

and such that $\Theta\left(x\right) = v_x$.

Let D be a regular smooth distribution on a manifold M. Fix a point x at M and consider a basis $\{v_x^i\}_i$ of tangent space $T_x M$. Then, for each v_x^i there exists a (local) vector field X^i tangent to the distribution D such that $\Theta^i(x) = v_x^i$. By using the inverse function theorem and shrinking (if it is necessary) the domain of the vector fields Θ^i, we prove that a regular distribution D is smooth if, and only if, D is locally finitely generated (see Definition B.0.21).
Following the inertia of this memory, we should define a kind of morphism to construct a category. Let D_1 and D_2 be two different distributions over the manifolds M_1 and M_2 respectively. A *morphism of distributions* from D_1 to D_2 is a pair of maps (Φ, ϕ), with $\Phi : D_1 \to D_2$ and $\phi : M_1 \to M_2$, commuting according to the diagram



where $\pi_Q : TQ \to Q$ defines the canonical projection of the tangent bundle of a manifold Q. Thus,

$$\pi_{M_2} \circ \Phi = \phi \circ \pi_{M_1}.$$

These kind morphisms permits us to construct the category of distributions which will be denoted by $\mathcal{D}$. Obviously, ϕ is characterized by Φ and, hence, the morphism will be sometimes denoted by Φ.
A morphism (Φ, ϕ) from D_1 to D_2 is said to be *smooth* if for any vector field Θ tangent to D_1 it satisfies that $\Phi \circ \Theta$ is a differentiable map from M_1 to TM_2. Thus, the category of smooth distributions, denoted by $\mathcal{SD}$, consists of the smooth distributions next to smooth morphisms of distributions.

A natural notion associated to the concept of distribution is the following one.

Definition B.0.13. An *integral submanifold* of a distribution D on a manifold M is a connected immersed submanifold N of M such that for every $y \in N$ the tangent space T_yN is a vector subspace of D_y. An integral submanifold N is called *maximal* if it is not contained in any other integral submanifold and it is said to be of *maximum dimension* if its tangent space at every point y is exactly D_y.
We say that a smooth distribution D on a manifold M is an *integrable distribution* if every point of M is contained in a maximal integral submanifold of maximum dimension.

Example B.0.14. Let $(M, \mathcal{F})$ be a foliated manifold, then we may define the *tangent distribution associated to* $\mathcal{F}$, denoted by $D^{\mathcal{F}}$, given by the following correspondence

$$x \in M \mapsto D_x^{\mathcal{F}} = T_x\mathcal{F}(x)\,.$$

It follows directly from the local foliation property that the tangent distribution is a smooth distribution.
Let $f : (M, \mathcal{F}) \to (N, \mathcal{S})$ be a morphism of foliated manifolds. Then, the restriction of the tangent map Tf to $D^{\mathcal{F}}$ induces a smooth morphism of distributions between $D^{\mathcal{F}}$ and $D^{\mathcal{S}}$.

Example B.0.15. Let C be a family of local smooth vector fields on M such that its domains cover M. Then it gives rise to a smooth singular distribution D^C: for each point $x \in M$, D_x^C is the vector space spanned by the values at x of the vector fields of C whose domains contain to x. We say that D^C is generated by C.

Also, associated to this example we have the following definition.

Definition B.0.16. A distribution D is called *invariant with respect to a family of (local) smooth vector fields* C if it is invariant with respect to every element of C: if $\Theta \in C$ and $\varphi_t^\Theta : U_t \to U_{-t}$ denotes the local flow of Θ, then we have

$$T_x\varphi_t^\Theta\left(D_x\right) = D_{\varphi_t^\Theta(x)},\ \forall x \in M,$$

whenever φ_t^Θ is defined.

Finally, the following result, due to Stefan [86] and Sussmann [88], gives an answer to the following question: what are the conditions for a smooth singular distribution to be the tangent distribution of a singular foliation?.

Theorem B.0.17 (Stefan-Sussman)**.** *Let D be a smooth singular distribution on a smooth manifold M. Then the following three conditions are equivalent:*

(a) D is integrable.

(b) D is generated by a family C of smooth vector fields, and is invariant with respect to C.

(c) D is the tangent distribution $D^{\mathcal{F}}$ of a smooth singular foliation $\mathcal{F}$.

Proof. $(a) \Rightarrow (b)$
Let C be the family of all (local) vector fields which are tangent to D. The smoothness condition of D implies that D is generated by C. Furthermore, if Θ is an arbitrary vector field tangent to D, then D is invariant with respect to Θ. In fact, let x be an arbitrary point in M, and denote by $\mathcal{F}(x)$ the maximal invariant submanifold of maximum dimension which contains x.
Then by definition (the condition of maximum dimension), for every point $y \in \mathcal{F}(x)$ we have,

$$T_y\mathcal{F}(x) = D_y,$$

which implies that the vector field Θ restricted to $\mathcal{F}(x)$ is tangent to $\mathcal{F}(x)$, i.e., the restriction of Θ to $\mathcal{F}(x)$ gives rise to a vector field of $\mathcal{F}(x)$. In particular, the local flow φ_t^{Θ} of Θ can be restricted to $\mathcal{F}(x)$ (this fact follows from the maximality condition on $\mathcal{F}(x)$).
Finally, using that Θ is tangent to $\mathcal{F}(x)$ we have

$$T_x\varphi_t^{\Theta}(D_x) = T_x\varphi_t^{\Theta}(T_x\mathcal{F}(x)) = T_{\varphi_t^{\Theta}(x)}\mathcal{F}(x) = D_{\varphi_t^{\Theta}(x)}.$$

$(b) \Rightarrow (c)$
Suppose that D is generated by a family C of smooth vector fields, and is invariant with respect to C. Let x be an arbitrary point of M, denote by

k the dimension of D_x, and choose k vector fields $\Theta_1, \ldots, \Theta_k$ of C such that $\Theta_1(x), \ldots, \Theta_k(x)$ span D_x. Denote by $\varphi_t^{\Theta_1}, \ldots, \varphi_t^{\Theta_k}$ the local flow of $\Theta_1, \ldots, \Theta_k$ respectively. The map

$$(t_1, \ldots, t_k) \mapsto \varphi_{t_1}^{\Theta_1} \circ \ldots \circ \varphi_{t_k}^{\Theta_k}(x), \tag{B.2}$$

is a local diffeomorphism from a k-dimensional disk to a k-dimensional submanifold containing x in M. The invariance of D with respect to C implies that this submanifold is an integral submanifold of maximum dimension. Gluing these local integral submanifolds together (wherever they intersect), we obtain a partition of M into a disjoint union of connected immersed integral submanifolds of maximum dimension, and thus, we may construct our foliation.
$(c) \Rightarrow (a)$
If $D = D^{\mathcal{F}}$ is the tangent distribution of a singular foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are maximal invariant submanifolds of maximum dimension for D. □

Definition B.0.18. An *involutive distribution* is a distribution D such that if Θ_1, Θ_2 are two arbitrary vector fields which are tangent to D, then their Lie bracket $[\Theta_1, \Theta_2]$ is also tangent to D.

Using the Stefan-Sussman theorem it is clear that if a singular distribution is integrable, then it is involutive. Conversely, for regular distributions we have:

Theorem B.0.19 (Frobenius)**.** *If a smooth regular distribution D is involutive then it is integrable, i.e., it is the tangent distribution of a regular foliation.*

Proof. We only have to use Eq. (B.2) again to prove this result. □

Note that, if in Frobenius theorem we omit the word regular, then it is false. As a counterexample we may construct the following distribution,

Example B.0.20. Consider the following singular distribution D on $\mathbb{R}^2$ given by

$$(x,y) \mapsto \begin{cases} T_{(x,y)}\mathbb{R}^2 & if \quad x > 0 \\ \langle \dfrac{\partial}{\partial x_{|(x,y)}} \rangle & if \quad x \leq 0 \end{cases}$$

Observe that D does not have constant dimension ($Dim\left(D_{(x,y)}\right) = 2$ if $x > 0$ and $Dim\left(D_{(x,y)}\right) = 1$ if $x \leq 0$). Furthermore, it is trivial that D is involutive.
Finally, for each $x > 0$, a maximal integral manifold of maximum dimension of D which contains to (x,y), for all $y \in \mathbb{R}$, is given by

$$\mathbb{R}^+ := \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$

Also, for each $(0, y_0)$, a maximal integral manifold of maximum dimension of D which contains to $(0, y_0)$, L, must verifies that

$$Dim\,(L) = 1, \quad T_{(0,y_0)}L = \langle \frac{\partial}{\partial x_{|(0,y_0)}} \rangle, \quad L \cap \mathbb{R}^+ = \emptyset.$$

Since that the point $(0, y_0)$ must belong to L, L has to be of the form

$$L := \{(x, y_0) \in \mathbb{R}^2 : x \leq 0\},$$

which is not possible since the leaf of D through a point is a manifold without boundary. Thus, D is smooth involutive but not integrable.

Now, in order to repair this problem we are going to give the following definition

Definition B.0.21. A smooth distribution D on a manifold M is called *locally finitely generated* if for any $x \in M$ there is a neighborhood U of x such that the $\mathcal{C}^{\infty}(U)$ −module of smooth tangent vector fields to D in U is finitely generated: there is a finite number of smooth vector fields $X_1, \ldots, X_k$ in U which are tangent to D, such that any smooth vector field Y in U which is tangent to D can be written as

$$Y = f_i X_i,$$

with $f_i \in \mathcal{C}^{\infty}(U)$.

So, the situation in the finitely generated case is better:

Theorem B.0.22 (Hermann)**.** *Any locally finitely generated smooth involutive distribution on a smooth manifold is integrable.*

See [1] to study this theorem in more detail.

Summary, Conclusions and Future works

In this thesis we have presented an algebraic/geometric approach to the study material bodies: (Lie) groupoids, Lie algebroids and smooth distributions are used in this context to characterize the uniformity, smooth uniformity and homogeneity of a material body.

Simple materials

For a simple body $\mathcal{B}$ we have considered two possibilities:

(1) $\Omega(\mathcal{B})$ is a Lie subgroupoid of the Lie groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$.

(2) $\Omega(\mathcal{B})$ is a just an algebraic subgroupoid of the Lie groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$.

In the first case, we characterize the homogeneity in terms of the material groupoid and algebroid. These results are equivalent to those obtained previously using $G-$structures [31].
In the second case, we have introduced the material distributions $A\Omega^T(\mathcal{B})$ to endow the material groupoid $\Omega(\mathcal{B})$ of a kind of "maximal pseudo-differentiable" structure which generalizes the structure of Lie subgroupoid of the Lie groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$.

Theorem B.0.23. *There exists a maximal foliation $\mathcal{F}$ of $\mathcal{B}$ such that all $X \in \mathcal{B}$, there is a unique transitive Lie subgroupoid $\Omega\left(\mathcal{F}\left(X\right)\right)$ of $\Pi^1\left(\mathcal{B}, \mathcal{B}\right)$ over $\mathcal{F}\left(X\right)$ consisting of material isomorphisms. Thus, any body $\mathcal{B}$ can be covered by a maximal foliation of smoothly uniform material submanifolds.*

This result proves the intuitive property of that any simple body $\mathcal{B}$ can be uniquely divided by a maximal foliation of (smoothly) uniform generalized sub-bodies.

Graded uniformity is presented as a "measure" of uniformity. Roughly speaking, $\mathcal{B}$ is uniform of grade p at a particle X if there exists a maximal smoothly uniform generalized sub-body of $\mathcal{B}$ which contains X. In this sense, a simple body $\mathcal{B}$ is smoothly uniform if, and only if, $\mathcal{B}$ is uniform of grade 3 at all the particles.

A notion of homogeneity for non-uniform bodies is also introduced by using the material distributions and its associated material foliations. In a purely intuitive sense, a simple body is homogeneous if all the leaves of its unique material foliation given in theorem B.0.23 are homogeneous and (locally) all these leaves can be straightened at the same time. This homogeneity is characterized in several different ways mimicking the known homogeneity. We used some of these characterizations on particular examples.

Pure mathematical results

We generalize this development to the context of general groupoids. In particular, for an algebraic subgroupoid $\overline{\Gamma}$ of a Lie groupoid $\Gamma \rightrightarrows M$. So, we construct the so-called material distributions: $A\overline{\Gamma}^T$ over Γ and $A\overline{\Gamma}^\sharp$ over M. These distributions turn out to be integrable and, therefore, there are foliations $\overline{\mathcal{F}}$ and $\mathcal{F}$ integrating $A\overline{\Gamma}^T$ and $A\overline{\Gamma}^\sharp$ respectively.

These distributions endow the groupoid $\overline{\Gamma}$ of a "pseudo-differentiable" structure which generalizes the structure of Lie subgroupoid of the Lie groupoid $\Gamma \rightrightarrows M$. Here, we use these tools in the pair groupoid obtaining the following interesting theorem:

Theorem B.0.24. *Let M be a manifold and N be a subset of M. Then, there exists a maximal foliation $\mathcal{F}$ of M such that N is union of leaves.*

With this, any subset of a smooth manifold can be endowed with a differentiable structure generalizing the structure of manifold. Indeed, N is a submanifold of M if, and only if, the foliation described in theorem B.0.24 has one unique leaf contained in N equal to N.

Cosserat media

Consider $F\mathcal{B}$ a Cosserat medium such that the second-order non-holonomic material groupoid $\overline{\Omega}(\mathcal{B})$ is a Lie subgroupoid of the Lie groupoid $\tilde{J}^1(F\mathcal{B})$. Then, we have given a new definition of homogeneity of the Cosserat continuum. This definition is characterized by using the second-order non-holonomic material groupoid and its associated second-order non-holonomic material algebroid $A\overline{\Omega}(\mathcal{B})$ in the following way:

Proposition B.0.25. *Let $\mathcal{B}$ be a Cosserat continuum. If $\mathcal{B}$ is homogeneous, then, $\overline{\Omega}(\mathcal{B})$ (resp. $A\overline{\Omega}(\mathcal{B})$) is an integrable non-holonomic prolongation of second order. Conversely, $\overline{\Omega}(\mathcal{B})$ (resp. $A\overline{\Omega}(\mathcal{B})$) is an integrable non-holonomic prolongation of second order implies that $\mathcal{B}$ is locally homogeneous.*

This result is translated to the context of covaraint derivatives obtaining that the Cosserat continuum is homogeneous if can be covered by some kind of "integrable" covariant derivatives.

Proposition B.0.26. *Let $\mathcal{B}$ be a Cosserat continuum. $\mathcal{B}$ is locally homogeneous if, and only if, for each material particle $X \in \mathcal{B}$ there exists a local coordinate system $\left(x^i\right)$ over $\mathcal{U} \subseteq \mathcal{B}$ with $X \in \mathcal{U}$ such that the local non-holonomic covariant derivative of second order ∇ satisfies*

(i) $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = R^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla_{\frac{\partial}{\partial x^k}} \dfrac{\partial}{\partial x^i_j} = \dfrac{\partial P^l_i}{\partial x^k} \dfrac{\partial}{\partial x^l_j}$

where

$$R^k_{l,ij} = \frac{\partial^2 P^k_l}{\partial x^j \partial x^i} + \frac{\partial^2 P^k_l}{\partial x^j \partial x^i},$$

takes values in $\mathcal{D}\left(A\overline{\Omega}\left(\mathcal{B}\right)\right)$.

Finally, we relate this definition with the known one respect to second-order non-holonomic $\overline{G}-$structures.

Proposition B.0.27. *A Cosserat continuum* $\mathcal{B}$ *is locally homogeneous if, and only if, there exists a reference crystal* $\overline{Z}^2_0$ *such that* $\mathcal{B}$ *is homogeneous (in the sense of second-order non-holonomic* $\overline{G}-$*structure) with respect to* $\overline{Z}^2_0$.

Future work

Thus, we conclude that, by introducing new tools to the constitutive theory of materials we have obtained results describing the structure of non-uniform bodies. The above results permit us to explore many ways for future researches. We shall mention here a few among them;

> **Cosserat media**. In section 3.2 we have studied new characterizations of homogeneity of the so-called Cosserat media [52] which is a model which may be applied to materials with microstructure like bones of rocks. However, we have always worked with the homogeneity for smoothly uniform materials. So, from the beginning we were assuming that the second-order non-holonomic material groupoid is a Lie subgroupoid of the second-order non-holonomic groupoid.
> Therefore, a new goal of research is to study how could we implement the notion of material distributions to this context in order to give a general treatment of this kind of materials.

Polar theories and higher-grade materials. For simple bodies the first gradient of a deformation at a particle is enough for the description of the material response of the body. But theories like *Polar Field theory* (see [44] and [11] for some application) add the necessity of higher-order gradients.
In general, an elastic material is completely characterized by constitutive relations which satisfy that, at each particle, depends only on the value of the successive gradients of the deformation at the particle. The order of the highest gradient which appears in the description is called the *grade* of the elastic material. In this sense, we may again define a material groupoid as the collection of all material isomorphisms. Therefore, the general theory developed in [51] could be applied to this case.

Global theory of materials. The space of configurations of a material body $\mathcal{B}$ has a (non unique) structure of differentiable manifold of infinite dimension [49, 58]. Therefore, one could try to define a mechanical response on this space which generalizes the structure of simple body and higher-grade body. With this, an infinite dimensional groupoid would be defined to make a similar development extending the others.

Material evolution. Material evolution can be considered as the time-like counterpart of material uniformity. In particular, instead of comparing two points at the same instant of time, the theory of material evolution concerns itself with the comparison of the point along the time [38].
In this case, the material groupoid of the material evolution may be defined as the largest groupoid which leaves the mechanical response invariant. Again, theory of groupoids, algebroids and distributions may be useful for the description of the time-like transformations of a material body.

Resumen, Conclusiones y Trabajos futuros

En esta tesis hemos presentado un enfoque algebraico/geométrico del estudio de los cuerpos materiales: grupoides (de Lie), algebroides de Lie y distribuciones diferenciables son utilizadas en este contexto para caracterizar la uniformidad, uniformidad diferenciable y homogeneidad de un cuerpo material.

Materiales simples

Para un cuerpo simple $\mathcal{B}$ hemos considerado dos posibilidades:

(1) $\Omega(\mathcal{B})$ es un subgrupoide de Lie del grupoide de Lie $\Pi^1(\mathcal{B}, \mathcal{B})$.

(2) $\Omega(\mathcal{B})$ is simplemente un subrupoide del grupoide de Lie $\Pi^1(\mathcal{B}, \mathcal{B})$.

En el primer caso, caracterizamos la homogeneidad en términos del grupoide material y el algebroide material. Esto resultados son equivalentes a aquellos obtenidos previamente usando $G-$estructuras [31].
En el segundo caso, hemos introducido las distribuciones materiales $A\Omega^T(\mathcal{B})$ para dotar al grupoide material $\Omega(\mathcal{B})$ de un tipo de estructura "maximal pseudo-diferenciable" que generaliza la estructura de subgrupoide de Lie del grupoide de Lie $\Pi^1(\mathcal{B}, \mathcal{B})$.

Theorem B.0.28. *Existe una foliación maximal* $\mathcal{F}$ *de* $\mathcal{B}$ *tal que para todo* $X \in \mathcal{B}$*, existe un único subgrupoide de Lie transitivo* $\Omega(\mathcal{F}(X))$ *de* $\Pi^1(\mathcal{B}, \mathcal{B})$ *sobre* $\mathcal{F}(X)$ *compuesto por isomorfismos materiales. Así, cualquier cuerpo* $\mathcal{B}$ *puede ser cubierto por una foliación maximal de subvariedades materiales diferenciablemente uniformes.*

Este resultado prueba la propiedad intuitiva de que cualquier cuerpo simple $\mathcal{B}$ puede ser únicamente dividido por una foliación maxiamal de sub-cuerpos generalizados (diferenciablemente) uniformes.

La uniformidad graduada es presentada como una "medida" de uniformidad. Así, $\mathcal{B}$ es uniforme de grado p en la parícula X si existe un sub-cuerpo generalizado maximal diferenciablemente uniforme de $\mathcal{B}$ que contiene a X. En este sentido, un cuerpo simple $\mathcal{B}$ es diferenciablemente uniforme si, y sólo si, $\mathcal{B}$ es uniforme de grado 3 en todas sus partículas.

Una noción de homogeneidad para cuerpos no-uniformes es también introducida usando las distribuciones materiales y sus foliaciones asociadas. En un sentido puramente intuitivo, un cuerpo simple es homogéneo si todas las hojas de su única foliación material dadas en el teorema B.0.28 son homogéneas y (localmente) todas estas hojas puede ser enderezadas al mismo tiempo. Esta homogeneidad es caracterizada de muchas formas imitando lo que ocurre con la homogeneidad conocida. Usamos estas caracterizaciones sobre ejemplos particulares.

Resultados puramente matemáticos

Generalizamos el desarrollo arriba explicado al contexto general de grupoides. En particular, para un subgrupoide algebraico $\overline{\Gamma}$ de un grupoide de Lie $\Gamma \rightrightarrows M$. Así, construimos las llamadas distribuciones materiales: $A\overline{\Gamma}^T$ sobre Γ y $A\overline{\Gamma}^\sharp$ sobre M. Estas distribuciones resultan ser integrables y, por lo tanto, hay foliaciones $\overline{\mathcal{F}}$ y $\mathcal{F}$ integrando $A\overline{\Gamma}^T$ y $A\overline{\Gamma}^\sharp$ respectivamente.

Estas distribuciones dotan al grupoide $\overline{\Gamma}$ de una estructura "pseudo-diferenciable" que generaliza la estructura de subgrupoide de Lie del grupoide de Lie $\Gamma \rightrightarrows M$. Aquí, usamos estas herramientas en el grupoide par obteniendo el siguiente teorema interesante:

Theorem B.0.29. *Sea M una variedad diferenciable y N un subconjunto de M. Entonces, existe una foliación maximal $\mathcal{F}$ de M tal que N es unión de hojas.*

Con esto, cualquier subconjunto de una variedad diferenciable puede ser dotado con una estructura diferenciable generalizando la estructura de variedad. De hecho, N es una subvariedad de M si, y sólo si, la foliación descrita en el teorema B.0.29 tiene una única hoja contenida en N igual a N.

Medios de Cosserat

Sea $F\mathcal{B}$ un medio de Cosserat tal que el grupoide material no-holónomo de segundo orden $\overline{\Omega}(\mathcal{B})$ es un subgrupoide de Lie del grupoide de Lie $\tilde{J}^1(F\mathcal{B})$. Así, hemos proporcionado una nueva definición de medio de Cosserat. Esta definición está caracterizada usando el grupoide material no-holónomo de segundo orden y su algebroide material no-holónomo de segundo orden asociado $A\overline{\Omega}(\mathcal{B})$ de la siguiente manera:

Proposition B.0.30. *Sea $\mathcal{B}$ un medio de Cosserat. Si $\mathcal{B}$ es homogéneo, entonces, $\overline{\Omega}(\mathcal{B})$ (resp. $A\overline{\Omega}(\mathcal{B})$) es una prolongación no-holónoma de segundo orden integrable. Recíprocamente, si $\overline{\Omega}(\mathcal{B})$ (resp. $A\overline{\Omega}(\mathcal{B})$) es una prolongación no-holónoma de segundo orden integrable, entonces, $\mathcal{B}$ es localmente homogéneo.*

Este resultado es transladado al contexto de derivadas covariantes obteniendo que el medio de Cosserat es homogéneo si puede ser cubierto por cierto tipo de derivada covariante "integrable".

Proposition B.0.31. *Sea $\mathcal{B}$ un medio de Cosserat. $\mathcal{B}$ es localmente homogéneo si, y sólo si, para cada partícula material $X \in \mathcal{B}$ existe un único sistema de coordenadas local $\left(x^i\right)$ sobre $\mathcal{U} \subseteq \mathcal{B}$ con $X \in \mathcal{U}$ tal que la derivada covariante no-holónoma de segundo orden ∇ dada por* *(i)* $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = R^k_{l,ij} \frac{\partial}{\partial x^k_l}$

(ii) $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i_j} = \frac{\partial P^l_i}{\partial x^k} \frac{\partial}{\partial x^l_j}$

donde

$$R^k_{l,ij} = \frac{\partial^2 P^k_l}{\partial x^j \partial x^i} + \frac{\partial^2 P^k_l}{\partial x^j \partial x^i},$$

toma valores en $\mathcal{D}\left(A\overline{\Omega}\left(\mathcal{B}\right)\right)$.

Finalmente, relacionamos esta definición con la conocida respecto $\overline{G}-$estructuras no-holónomas de segundo orden.

Proposition B.0.32. *Un medio de Cosserat* $\mathcal{B}$ *es localmente homogéneo si, y sólo si, existe una referencia* $\overline{Z}^2_0$ *tal que* $\mathcal{B}$ *es homogéneo (en el sentido de* $\overline{G}-$*estructuras no-holónomas de segundo orden) con respecto a* $\overline{Z}^2_0$.

Trabajo futuro

Así, concluimos que, introduciendo nuevas herramientas a la teoría constitutiva de materiales hemos obtenido resultados describiendo la estructura de cuerpos no-uniformes. Los resultados arriba explicados nos permiten explorar muchos caminos para investigaciones futuras. Mencionaremos aquí algunas de ellas;

Medios Cosserat. En la sección 3.2 hemos estudiado nuevas caracterizaciones de homogeneidad de los llamados medios de Cosserat [52] que es un modelo que puede ser aplicado a materiales con microestructura como huesos o rocas. Sin embargo, hemos trabajado siempre con la homogeneidad de materiales diferenciablemente uniformes. Así, desde el principio hemos asumido que el grupoide material no-holónomo de segundo orden es un subgrupoide de Lie del grupoide no-holónomo de segundo orden.
Por lo tanto, un nuevo objetivo de investigación es estudiar como podríamos implementar la noción de distribuciones materiales a este

contexto para dar un tratamiento general a este tipo de materiales.

Teorías Polares y materiales de alto grado. Para cuerpos simples el primer gradiente de una deformación en una partícula es suficiente para la descripción de la respuesta material del cuerpo. Pero teorías como la *Teoría de campos polares* (see [44] y [11] para alguna aplicación) añaden la necesidad de gradientes de un grado más alto.
En general, un material elástico está completamente caracterizado por las relaciones constitutivas que satisfacen que, en cada partícula, dependen del valor de sucesivos gradientes de la deformación en la partícula. El orden del gradiente más alto que aparece en la descripción es llamado *grado* del material elástico. En este sentido, podemos de nuevo definir un grupoide material como la colección de todos los isomorfismos materiales. Por lo tanto, la teoría general desarrollada en [51] podría ser aplicada en este caso.

Teoría global de materiales. El espacio de configuraciones de un cuerpo material $\mathcal{B}$ tiene una (no única) estructura de variedad diferenciable de dimensión infinita [49, 58]. Por lo tanto, uno podría intentar definir una respuesta mecánica sobre este espacio que generalice la de cuerpos simples y cuerpos de grado alto. Con esto, un grupoide infinito dimensional sería definido para hacer un desarrollo similar extendiendo los otros.

Evolución material. La evolución material puede ser considerada como una contraparte de la uniformidad material. En particular, en lugar de comparar dos puntos en el mismo instante de tiempo, la teoría de evolución material nos permite comparar dos puntos en diferentes instantes de tiempo [38].
En este caso, el grupoide material del material de evolución puede ser definido como el grupoide más grande que deja la respuesta mecánica invariante. De nuevo, las teoría de grupoides, algreboides y distribuciones pueden ser útiles en la descripción de las transformaciones temporales del cuerpo material.

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